Annals of Pure and Applied Logic 48 (1990) 135–196
North-Holland


# DOMAIN INTERPRETATIONS OF MARTIN-LÖF'S PARTIAL TYPE THEORY

Erik PALMGREN and Viggo STOLTENBERG-HANSEN
*Department of Mathematics, Uppsala University, Uppsala, Sweden*

Communicated by D. van Dalen
Received 2 February 1989

Martin-Löf's intuitionistic type theory may be seen as an open system or set of rules formalizing part of constructive reasoning and mathematics. The system can be viewed or read in several ways, in particular as a set theory and as a logic. It may also be considered as a programming language, as described in Martin-Löf [7] and Nordström et al. [11]. There is what is called a standard interpretation of type theory given by Martin-Löf. This interpretation differs from the kind of interpretations or models many of us are used to, following Tarski and using set theory. Instead the rules are explained, i.e. given meaning or content, at the same time as they are being presented.

The system presented in Martin-Löf [9], or variants thereof, is often referred to as total type theory. Martin-Löf has extended total type theory to partial type theory by introducing an iteration type $\Omega$ and an element $\omega \in \Omega$ (see Section 4 below) which, for example, allowed him to define a fixed point operator within the theory. Partial type theory turns out to be inconsistent when considered as a logic in the sense that each type is non-empty, including the type $N_0$ for absurdity, which hence is true. Nonetheless, partial type theory is a perfectly good set theory and programming language. Unable to give a satisfactory standard interpretation of partial type theory, Martin-Löf [8] gave a domain interpretation in which each expression for an element of a type (this we call a term) is interpreted into a fixed domain. A presentation of this interpretation using information systems is given in Eklund [5].

In this paper we give an interpretation of partial type theory using the theory of Scott–Ershov domains and continuous functors from a domain into a category of domains. Each expresssion for a type, which we call set in the formal system, is interpreted as a continuous functor and hence as a continuous family of domains, and each term, i.e. expression for an element of a type, is interpreted as a continuous function over the continuous family of domains given by the type of the element. Dependent types in contexts are interpreted by what turns out to be a special case of Grothendieck fibrations. Our interpretation is effective (in the sense of computability) in that the interpretation of a type is an effective continuous functor and the interpretation of a term is an effective continuous function. The interpretation has subsequently been extended to include universes in Palmgren [22].

There are other interpretations of type theory in the literature. Aczel [1] and Smith [18] give a logical interpretation of (total) type theory into a type-free theory of propositions. Beeson [2] gives a realizability interpretation of (total) type theory, Lindström [6] gives cpo interpretations of partial type theory within type theory itself, and Rezus [16] gives an interpretation into the graph model $P\omega$ of total type theory with one universe but lacking identity types. Also related is Coquand et al. [3], [4] where, primarily, models for the polymorphic $\lambda$-calculus are considered. They independently arrive at the same basic concepts such as continuous functors and Grothendieck fibrations but in a more general setting. The notion of a continuous functor over a domain, or at least a cpo, appears already in Plotkin [15].

The origin for this paper is a seminar by Dag Normann where he presented his work on what he calls Kleene spaces, see [12]. In contrast to the elements of domains, the elements of Kleene spaces are in some sense total. Thus it is not surprising that the notion of a continuous family of Kleene spaces, or a parametrization, is extremely and perhaps unnecessarily complicated. Our original objective was to do for domains what Normann tried to do for his Kleene spaces. This is carried out in Palmgren [13], where many of the basic notions of this paper are introduced and studied.

In Section 1 we introduce the concept of a continuous family of domains indexed by a domain, also called a parametrization. Then, in Section 2, we study operations on parametrizations needed in order to carry out our interpretations. In Section 3 we introduce the notion of an effective parametrization and show that all the concepts and operations of Sections 1 and 2 have their effective counterparts. Our version of partial intuitionistic type theory is presented in Section 4. Then our first domain interpretation is carried out in Section 5. It satisfies the $\eta$-rule but it is not adequate for the operational semantics of type theory. Therefore this interpretation is modified in Section 6 and the modified version is proved to be adequate for the operational semantics.

We wish to thank Per Martin-Löf for his help to increase our understanding of intuitionistic type theory, through seminars and informal conversations. In particular, we believe he was the first to suggest interpreting the $I$-type using initial segments of domains. After the completion of this paper, the abstract Martin-Löf [10] was brought to our attention, indicating that Martin-Löf more than anticipated many of our results. We also thank Dag Normann for the reason already mentioned and Ed Griffor and Ingrid Lindström for many stimulating and helpful conversations on the topic of this paper.

## 1. Continuous families of domains

In this section we first review some basic notions of domain theory. Then we make precise the category of domains, DOM, that we shall consider. A continuous family of domains indexed by a fixed domain $D$ will be a continuous functor from $D$ into DOM, where $D$ is considered as a category in the usual way. We define the notion of continuity for a functor in this special setting, motivated by continuity for domains. It turns out that this definition is equivalent to the usual notion of continuity for functors. Finally we observe that the continuity of functors is preserved under the usual operations on domains such as the function space operation. We will omit most proofs in this section. They can easily be obtained from some basic references such as Scott [17] and Smyth and Plotkin [19].

Let $D = (D; \sqsubseteq, \perp)$ be a partially ordered set with least element $\perp$. A set $A \subseteq D$ is *directed* if $A \neq \emptyset$ and if $x, y \in A$ then there is $z \in A$ such that $x \sqsubseteq z$ and $y \sqsubseteq z$. $D$ is a *complete partial order* (*cpo*) if every directed set $A \subseteq D$ has a least upper bound in $D$, denoted $\bigsqcup A$. An element $a \in D$ is said to be *compact* or finite if whenever $a \sqsubseteq \bigsqcup A$, where $A$ is directed, then there is $x \in A$ such that $a \sqsubseteq x$. Let $D_c = \{a \in D: a \text{ compact}\}$, and for each $x \in D$ let $\text{approx}(x) = \{a \in D_c : a \sqsubseteq x\}$, the set of compact approximations of $x$. A cpo $D$ is *algebraic* if for each $x \in D$, $\text{approx}(x)$ is directed and $x = \bigsqcup \text{approx}(x)$. $D$ is *consistently complete* if whenever $a, b \in D_c$ are *consistent*, i.e. $\{a, b\}$ is bounded from above in $D$, then $\bigsqcup \{a, b\} = a \sqcup b$ exists in $D$. A *Scott–Ershov domain* or simply a *domain* is a consistently complete algebraic cpo.

A function $f: D \to E$ between domains is *monotone* if $x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$, and *continuous* if, in addition, $f(\bigsqcup A) = \bigsqcup f(A)$ for every directed set $A$. It is easily seen that $f$ is continuous if and only if $f$ is monotone and $f(x) = \bigsqcup \{f(a) : a \in \text{approx}(x)\}$ for each $x \in D$. Furthermore, each monotone $f: D_c \to E$ has a unique continuous extension $\bar{f}: D \to E$, namely

$$\bar{f}(x) = \bigsqcup \{f(a) : a \in \text{approx}(x)\}.$$

The following simple characterization of continuity is often useful.

**Lemma 1.1.** *A function $f: D \to E$ between domains is continuous if and only if $f$ is monotone and whenever $b \in \text{approx}(f(x))$ then there is $a \in \text{approx}(x)$ such that $b \sqsubseteq f(a)$.*

Let $D$ and $E$ be domains. Then the *cartesian product* $D \times E$ is a domain when ordered coordinatewise, and the compact elements are $(D \times E)_c = D_c \times E_c$. The *separated sum* $D + E$ is the disjoint union of $D$ and $E$ adjoined with a element $\perp$,

$$D + E = \{\perp\} \cup \{(0, x) : x \in D\} \cup \{(1, y) : y \in E\}.$$

With the ordering inherited from $D$ and $E$ and with $\perp$ as the least element, $D + E$ is a domain. The *lift* $D_\perp$ of $D$ is the domain obtained by adjoining a new bottom element $\perp$ to $D$. The *function space* $D \to E$ consists of all continuous functions from $D$ to $E$. It is also a domain with the pointwise ordering, i.e.

$$f \sqsubseteq g \quad \text{iff} \quad \forall x \in D\, f(x) \sqsubseteq g(x).$$

For the sake of notation and future reference we describe the compact elements of $D \to E$. Let $a \in D_c$ and $b \in E_c$ and define $\langle a; b\rangle : D \to E$ by

$$\begin{aligned}\langle a; b\rangle(x) &= b \quad \text{if } a \sqsubseteq x,\\ &= \perp \quad \text{else.}\end{aligned}$$

Then $\langle a; b\rangle$ is the least element in $D \to E$ which maps $a$ to $b$. Furthermore $\langle a; b\rangle$ is compact and the compact elements of $D \to E$ are precisely the suprema of consistent finite sets of such functions.

Let $D$ and $E$ be domains. Then $D$ is a *subdomain of* $E$, denoted by $D \lhd E$, if $D = (D; \sqsubseteq, \perp)$ is a substructure of $E = (E; \sqsubseteq, \perp)$ and the following hold:

(i) $A$ is a directed subset of $D \Rightarrow \bigsqcup_E A \in D$,

(ii) $D_c \subseteq E_c$, and

(iii) $\{a, b\} \subseteq D_c$ consistent in $E \Rightarrow a \sqcup_E b \in D$.

Let $E$ be a domain and let $\{D_i : i \in I\}$ be a directed set of subdomains of $E$ using the order $\lhd$. Then we define a closure operator by

$$\mathrm{cl}(\{D_i : i \in I\}) = \{\textstyle\bigsqcup_E C : C \subseteq \bigcup\{(D_i)_c : i \in I\},\ C \text{ directed}\}.$$

**Proposition 1.2.** *If $\{D_i : i \in I\}$ is a directed set of subdomains of $E$ then $\mathrm{cl}(\{D_i : i \in I\}) \lhd E$. In fact $\mathrm{cl}(\{D_i : i \in I\})$ is the supremum of $\{D_i : i \in I\}$ in the set of subdomains of $E$.*

This proposition is the main ingredient in the proof of the following theorem due to Scott [17].

**Theorem 1.3.** *Let $E$ be a domain and let $P_E = \{D : D \lhd E\}$. Then $P_E = (P_E; \lhd, \{\perp\})$ is a domain.*

For rather trivial set-theoretic reasons it does not suffice to use the domain of subdomains of a given domain in order to define continuity of families of domains (cf. Proposition 1.12). We need the more general situation where we consider embeddings of one domain into another. This is best described using the notion of projection pairs.

A *projection pair* $(f, g)$ from a domain $D$ to a domain $E$ is a pair of continuous functions $f : D \to E$ and $g : E \to D$ such that $g \circ f = \mathrm{id}_D$, the identity function on $D$, and $f \circ g \sqsubseteq \mathrm{id}_E$. The importance of this notion lies in the fact that the existence of a projection pair $(f, g)$ from $D$ to $E$ is equivalent to $D$ being isomorphic to a subdomain of $E$. In fact, $f$ is injective and the subdomain is $f(D)$. Normally we write a projection pair as $(f^+, f^-)$ where $f^+ : D \to E$ is the *embedding* and $f^- : E \to D$ is the *projection*.

For later reference we state some easily verified properties of projection pairs.

**Lemma 1.4.** *Let $(f^+, f^-)$ be a projection pair from $D$ to $E$. Then*

(i) $f^+(\perp_D) = \perp_E$ *and* $f^-(\perp_E) = \perp_D$,

(ii) $f^+$ *is orderpreserving, i.e.* $x \sqsubseteq y \Leftrightarrow f^+(x) \sqsubseteq f^+(y)$,
(iii) *for each* $x \in D$, $y \in E$, $f^+(x) \sqsubseteq y$ *iff* $x \sqsubseteq f^-(y)$,
(iv) $a \in D_c \Rightarrow f^+(a) \in E_c$, *and*
(v) *if* $(g^+, g^-)$ *is another projection pair from* $D$ *to* $E$ *then* $f^+ \sqsubseteq g^+$ *iff* $g^- \sqsubseteq f^-$.

Note that (v) implies that one half of a projection pair uniquely determines the other half. Thus one may just consider embeddings, which sometimes is simpler. The following proposition characterizes when a function is an embedding.

**Proposition 1.5.** *Let* $D$ *and* $E$ *be domains and suppose* $f : D_c \to E_c$ *satisfies*
(a) $f(\perp_D) = \perp_E$,
(b) $f$ *is orderpreserving, and*
(c) *if* $f(a)$ *and* $f(b)$ *are consistent in* $E$ *then* $a$ *and* $b$ *are consistent in* $D$ *and*

$$f(a \sqcup b) = f(a) \sqcup f(b).$$

*Then the unique continuous extension* $f^+ : D \to E$ *of* $f$ *is the embedding of a projection pair from* $D$ *to* $E$.

The conditions of the proposition are clearly necessary.

Any partially ordered set may be viewed as a category. For a domain $D$, this is done by letting the objects of the category be the elements of $D$ and letting the morphism set between objects $x$ and $y$ be the one point set $\{i_x{}^y\}$ precisely when $x \sqsubseteq y$ and the empty set otherwise.

The *category of domains* that we consider we call DOM. The class of objects of DOM is the class of all domains. The morphism set from a domain $D$ to a domain $E$ is the set of all projection pairs from $D$ to $E$. Suppose $(f^+, f^-)$ is a projection pair from $D$ to $E$ and $(g^+, g^-)$ is a projection pair from $E$ to $F$. Then the composition of $(f^+, f^-)$ and $(g^+, g^-)$ is defined by

$$(g^+, g^-) \circ (f^+, f^-) = (g^+ \circ f^+, f^- \circ g^-).$$

It is easily verified that the composition of two projection pairs is again a projection pair and that DOM is a category.

Let $F : D \to \mathrm{DOM}$ be a functor. Recall that this means that $F(x)$ is a domain for each $x \in D$, and if $x \sqsubseteq y$ then $F(i_x{}^y)$ is a projection pair from $F(x)$ to $F(y)$ such that $F(i_x{}^x) = \mathrm{id}_{F(x)}$, and if $x \sqsubseteq y$ and $y \sqsubseteq z$ then $F(i_x{}^z) = F(i_y{}^z) \circ F(i_x{}^y)$. When $x \sqsubseteq y$ we shall use the notation $(F^+_{xy}, F^-_{xy})$ or, when that notation becomes too cumbersome, $(F^+[x, y], F^-[x, y])$, for $F(i_x{}^y)$. In case the functor is evident from the context, we will write $z^{(y)}$ for $F^+_{xy}(z)$, and $w_{(x)}$ for $F^-_{xy}(w)$ when $z \in F(x)$ and $w \in F(y)$. The latter notation is the most convenient one, not duplicating information unnecessarily.

A functor $F : D \to \mathrm{DOM}$ may be viewed as a family of domains indexed by the domain $D$. Such a family is clearly monotone in the sense that if $x \sqsubseteq y$ in $D$ then there is a projection pair from $F(x)$ to $F(y)$, that is $F(x)$ is isomorphic to a subdomain of $F(y)$. Along with Proposition 1.2 and Theorem 1.3, this motivates the following notion of continuity.

**Definition 1.6.** Let $F: D \to \mathrm{DOM}$ be a functor. Then $F$ is *continuous* if for each directed set $A \subseteq D$, $F(\bigsqcup A) = \mathrm{cl}(\{F^+[y, \bigsqcup A](F(y)) : y \in A\})$. A continuous functor $F: D \to \mathrm{DOM}$ is called a *parametrization*.

We use the term parametrization for continuous functors in our special setting just to remind us of this setting. There are several equivalent characterizations of continuity.

**Theorem 1.7.** *Let $F: D \to \mathrm{DOM}$ be a functor. Then the following are equivalent.*
(i) *$F$ is continuous.*
(ii) *For each $x \in D$, $F(x) = \mathrm{cl}(\{F^+_{ax}(F(a)) : a \in \mathrm{approx}(x)\})$.*
(iii) *For each $x \in D$, $F(x) \cong \varinjlim\{F(a) : a \in \mathrm{approx}(x)\}$, where $\varinjlim$ denotes the directed colimit in* DOM.

In the following proposition we prove yet another characterization of continuity, one which will frequently be used in the sequel.

**Proposition 1.8.** *Let $F: D \to \mathrm{DOM}$ be a functor. Then $F$ is continuous if and only if for each $x \in D$,*

$$F(x)_c = \bigcup \{F(a)_c^{(x)} : a \in \mathrm{approx}(x)\}.$$

**Proof.** First assume that $F$ is continuous. By Lemma 1.4(iv) we need only show the inclusion $\subseteq$. Suppose $b \in F(x)_c$. By Theorem 1.7, $b = \bigsqcup C$ where $C \subseteq \bigcup \{F(a)_c^{(x)} : a \in \mathrm{approx}(x)\}$ and $C$ is directed. Since $b$ is compact, there is $d \in C$ such that $b \sqsubseteq d$. But then $b = d$ and hence there is $a \in \mathrm{approx}(x)$ such that $b \in F(a)_c^{(x)}$.

For the converse, assume the condition is true. We show that for each $x \in D$, $F(x) = \mathrm{cl}(\{F(a)^{(x)} : a \in \mathrm{approx}(x)\})$. By Proposition 1.2, we again need only show one inclusion. So suppose $y \in F(x)$. Then $\mathrm{approx}_{F(x)}(y) \subseteq \bigcup \{F(a)_c^{(x)} : a \in \mathrm{approx}(x)\}$ by the condition and hence

$$y = \bigsqcup \mathrm{approx}_{F(x)}(y) \in \mathrm{cl}(\{F(a)^{(x)} : a \in \mathrm{approx}(x)\}). \quad \square$$

**Proposition 1.9.** *Let $F: D \to \mathrm{DOM}$ be a functor. If $x \sqsubseteq y \sqsubseteq z$ then $F(x)_c^{(z)} \subseteq F(y)_c^{(z)}$.*

**Proof.** Suppose $b \in F(x)_c^{(z)}$ and let $b_1 \in F(x)_c$ such that $b_1^{(z)} = b$. Then

$$b = b_1^{(z)} = b_1^{(y)(z)}.$$

But $b_1^{(y)} \in F(y)_c$ since $F^+_{xy}$ is an embedding. $\square$

We now give some simple but important examples of parametrizations of domains. It will turn out that essentially these examples combined with certain closure operations are what we need for our interpretation.

**Example 1.10.** *Constant parametrizations.* Let $D$ and $E$ be domains and let $F: D \to \mathrm{DOM}$ be defined by $F(x) = E$ for each $x \in D$, and $F(i_x{}^y) = (\mathrm{id}_E, \mathrm{id}_E)$ whenever $x \sqsubseteq y$.

**Example 1.11.** *Initial segments of domains.* Let $D$ be a domain and for each $x \in D$ let $D^x = \{y \in D : y \sqsubseteq x\}$. It is easily verified that $D^x$ is a domain with the inherited ordering, in fact $D^x$ is a subdomain of $D$ and $D_c \cap D^x = (D^x)_c$. Define $F: D \to \mathrm{DOM}$ by $F(x) = D^x$, and if $x \sqsubseteq y$ define $F^+_{xy}: D^x \to D^y$ by $F^+_{xy}(z) = z$ and $F^-_{xy}: D^y \to D^x$ by $F^-_{xy}(z) = \bigsqcup(\mathrm{approx}(z) \cap D^x)$. Clearly $(F^+_{xy}, F^-_{xy})$ is a projection pair and $F$ is a functor. $F$ is continuous by Proposition 1.8.

It is easily seen that the parametrizations are closed under the cartesian product, separated sum and lifting. Parametrizations are also closed under the function space construction, which is a special case of Theorem 2.11. To be precise, suppose $F: D \to \mathrm{DOM}$ and $G: D \to \mathrm{DOM}$ are continuous functors. Define $(F \times G): D \to \mathrm{DOM}$ by $(F \times G)(x) = F(x) \times G(x)$ and, when $x \sqsubseteq y$, $(F \times G)^+_{xy}(z, w) = (F^+_{xy}(z), G^+_{xy}(w))$. (Recall that it suffices to define half of a projection pair.) For the separated sum define $(F + G): D \to \mathrm{DOM}$ by $(F + G)(x) = F(x) + G(x)$ and, when $x \sqsubseteq y$,

$$
\begin{aligned}
(F+G)^+_{xy}(w) &= \perp_{F(y)+G(y)} && \text{if } w = \perp, \\
&= (0, F^+_{xy}(\pi_1(w))) && \text{if } \pi_0(w) = 0, \\
&= (1, G^+_{xy}(\pi_1(w))) && \text{if } \pi_0(w) = 1.
\end{aligned}
$$

For lifting, define $F_\perp: D \to \mathrm{DOM}$ by $F_\perp(x) = F(x)_\perp$ and, when $x \sqsubseteq y$,

$$
\begin{aligned}
(F_\perp)^+_{xy}(w) &= \perp_{F(y)_\perp} && \text{if } w = \perp_{F(y)_\perp}, \\
&= F^+_{xy}(w) && \text{if } w \neq \perp_{F(y)_\perp}.
\end{aligned}
$$

Finally define $(F \to G): D \to \mathrm{DOM}$ by $(F \to G)(x) = F(x) \to G(x)$ and, when $x \sqsubseteq y$, define $(F \to G)^+_{xy}: (F(x) \to G(x)) \to (F(y) \to G(y))$ by

$$(F \to G)^+_{xy}(f) = G^+_{xy} f F^-_{xy}.$$

**Proposition 1.12.** *Let $F: D \to \mathrm{DOM}$ and $G: D \to \mathrm{DOM}$ be continuous functors. Then $(F \times G)$, $(F + G)$, $F_\perp$ and $(F \to G)$ are continuous functors.*

Finally we show that parametrizations are closed under substitutions of continuous functions.

**Proposition 1.13.** *Let $F: E \to \mathrm{DOM}$ be a continuous functor and let $f: D \to E$ be a continuous function. Then $G: D \to \mathrm{DOM}$ is a continuous functor, where $G(x) = F(f(x))$ for $x \in D$, and $G(i_x{}^y) = F(i_{f(x)}{}^{f(y)})$ when $x \sqsubseteq y$.*

**Proof.** By the monotonicity of $f$ it follows that $G$ is a functor. Suppose $b \in G(x)_c = F(f(x))_c$. Then, by the continuity of $F$, there is $d \in \mathrm{approx}_E(f(x))$ such that $b \in F(d)_c{}^{(f(x))}$. By the continuity of $f$, there is $a \in \mathrm{approx}(x)$ such that $d \sqsubseteq f(a)$. But then, by Proposition 1.9, $b \in F(f(a))_c{}^{(f(x))} = G(a)_c{}^{(x)}$. $\square$

## 2. Operations on parametrizations

In this section we study operations on continuous families of domains, or parametrizations, needed for our interpretation of intuitionistic type theory.

Let $F: D \to \mathrm{DOM}$ be a continuous functor. Define

$$\Sigma(D, F) = \{(x, y): x \in D, y \in F(x)\}$$

and order $\Sigma(D, F)$ by

$$(x, y) \sqsubseteq (z, w) \quad \Leftrightarrow \quad x \sqsubseteq_D z \quad \text{and} \quad y^{(z)} \sqsubseteq_{F(z)} w.$$

The partially ordered set $\Sigma(D, F)$ is the *disjoint sum of $F$ over $D$*.

**Theorem 2.1.** *Let $F: D \to \mathrm{DOM}$ be a continuous functor. Then $\Sigma(D, F)$ is a domain.*

**Proof.** For ease of notation we write $\Sigma$ for $\Sigma(D, F)$. It easily verified that $\Sigma$ is a partially ordered set with a least element $(\bot, \bot_{F(\bot)})$. Suppose $A \subseteq \Sigma$ is a directed set. Then $\pi_0(A) = \{x \in D: \exists y\ (x, y) \in A\}$ is directed and hence $w = \bigsqcup \pi_0(A) \in D$. Let $\pi_1(A) = \{y^{(w)}: (x, y) \in A\}$. Then $\pi_1(A)$ is a directed subset of $F(w)$ and hence $(\bigsqcup \pi_0(A), \bigsqcup \pi_1(A)) \in \Sigma$. We claim that $\bigsqcup A = (\bigsqcup \pi_0(A), \bigsqcup \pi_1(A))$. It is clearly an upper bound. Suppose $(u, v) \in \Sigma$ is an upper bound of $A$. Then for each $x \in \pi_0(A)$, $x \sqsubseteq u$ so $w = \bigsqcup \pi_0(A) \sqsubseteq u$. Let $y \in \pi_1(A)$. Then $y = y_1{}^{(w)}$ for some $(x, y_1) \in A$ and hence $y^{(u)} = y_1{}^{(w)(u)} = y_1{}^{(u)} \sqsubseteq v$ since $(x, y_1) \sqsubseteq (u, v)$. But then $(\bigsqcup \pi_0(A), \bigsqcup \pi_1(A)) \sqsubseteq (u, v)$ so $\bigsqcup A = (\bigsqcup \pi_0(A), \bigsqcup \pi_1(A))$.

Next we show that $\Sigma_c = \{(a, b): a \in D_c, b \in F(a)_c\}$. Suppose $(a, b) \in \Sigma_c$. Let $A_1 = \{(a, c): c \in \mathrm{approx}_{F(a)}(b)\}$. Then $A_1$ is a directed set and $\bigsqcup A_1 = (a, b)$, so $b \in F(a)_c$. Similarly, let $A_0 = \{(d, \bot_{F(d)}): d \in \mathrm{approx(a)}\}$. Then $A_0$ is a directed set, so again $a \in D_c$. Conversely, suppose $a \in D_c$ and $b \in F(a)_c$, and let $A \subseteq \Sigma$ be a directed set such that $(a, b) \sqsubseteq \bigsqcup A$. By the above, $\bigsqcup A = (\bigsqcup \pi_0(A), \bigsqcup \pi_1(A))$ so $a \sqsubseteq \bigsqcup \pi_0(A)$ and $b^{(w)} \sqsubseteq \bigsqcup \pi_1(A)$, where $w = \bigsqcup \pi_0(A)$. Since $a \in D_c$. there is $x \in \pi_0(A)$ such that $a \sqsubseteq x$. Furthermore $b^{(w)} \in F(w)_c$, so there is $y \in \pi_1(A)$ such that $b^{(w)} \sqsubseteq y$. Let $y_1$ be such that $(x, y_1) \in A$ and let $(x', y') \in A$ be such that $y = y'^{(w)}$. Then choose $(x'', y'') \in A$ such that $(x, y_1) \sqsubseteq (x'', y'')$ and $(x', y')$

$\sqsubseteq(x'', y'')$. Thus $a \sqsubseteq x \sqsubseteq x''$. Furthermore,

$$b^{(x'')(w)} = b^{(w)} \sqsubseteq y = y'^{(w)} = y'^{(x'')(w)} \sqsubseteq y''^{(w)},$$

so $b^{(x'')} \sqsubseteq y''$. This means $(a, b) \sqsubseteq (x'', y'') \in A$.

Now we show that $\Sigma$ is an algebraic cpo. First we show that approx$(x, y)$ is a directed set, for $(x, y) \in \Sigma$. So let $(a, b), (c, d) \in$ approx$(x, y)$. Then $a' = a \sqcup c \in D_c$, since $D$ is consistently complete. Furthermore $b^{(a')}, d^{(a')} \in F(a')_c$ are consistent, since both are dominated by $y_{(a')}$. Let $b' = b^{(a')} \sqcup d^{(a')} \in F(a')_c$. Then

$$b'^{(x)} = (b^{(a')} \sqcup d^{(a')})^{(x)} = b^{(x)} \sqcup d^{(x)} \sqsubseteq y,$$

since embeddings preserve arbitrary suprema. Thus approx$(x, y)$ is directed. Recall that

$$\bigsqcup \text{approx}(x, y) = (\bigsqcup \pi_0(\text{approx}(x, y)), \bigsqcup \pi_1(\text{approx}(x, y))).$$

Clearly, $\pi_0(\text{approx}(x, y)) = \text{approx}(x)$, so $x = \bigsqcup \pi_0(\text{approx}(x, y))$. We claim that $\pi_1(\text{approx}(x, y)) = \text{approx}_{F(x)}(y)$. Firstly,

$$\pi_1(\text{approx}(x, y)) = \{b^{(x)} : a \in \text{approx}(x), b \in F(a)_c, b^{(x)} \sqsubseteq y\},$$

so $\pi_1(\text{approx}(x, y)) \subseteq \text{approx}_{F(x)}(y)$. For the converse inclusion suppose that $d \in \text{approx}_{F(x)}(y)$. Then by the continuity of $F$ there is $a \in \text{approx}(x)$, $b \in F(a)_c$ such that $b^{(x)} = d$. Thus equality holds and $y = \bigsqcup \pi_1(\text{approx}(x, y))$.

Finally, the argument to show that $\Sigma$ is consistently complete is similar to the argument showing that approx$(x, y)$ is directed. □

Note that the continuity of $F$ was used at precisely one point in the proof, namely in showing that $\Sigma(D, F)$ was algebraic.

In the special case that $F : D \to \text{DOM}$ is the constant parametrization of Example 1.10, i.e. $F(x) = E$ for each $x \in D$, then $\Sigma(D, F) = D \times E$.

In order to interpret $\Sigma$-formation in intuitionistic type theory as a parametrization, we must also consider $\Sigma$ as a functor.

**Definition 2.2.** Let $F : D \to \text{DOM}$ be a continuous functor. Then $G$ is a *parametrization over* $F$ if $G : \Sigma(D, F) \to \text{DOM}$ is a continuous functor.

Suppose $F : D \to \text{DOM}$ is a continuous functor and that $G$ is a parametrization over $F$. Fix $x \in D$ and consider $(y)G(x, y) : F(x) \to \text{DOM}$, where $(y)$ denotes abstraction in the variable $y$. Then $(y)G(x, y)$ is a continuous functor for each $x \in D$ by Proposition 1.13, since it is just the composition of $G$ with $f_x : F(x) \to \Sigma(D, F)$ defined by $f_x(y) = (x, y)$, which clearly is continuous. It follows that $\Sigma(F(x), (y)G(x, y))$ is a domain for each $x \in D$.

Define a functor $\Sigma(F, G) : D \to \text{DOM}$ by

$$\Sigma(F, G)(x) = \Sigma(F(x), (y)G(x, y))$$

and, when $x \sqsubseteq y$,

$$\Sigma(F, G)^+_{xy}(u, v) = (u^{(y)}, v^{(y,u^{(y)})}) \quad\text{and}\quad \Sigma(F, G)^-_{xy}(z, w) = (z_{(x)}, w_{(x,z_{(x)})}).$$

**Theorem 2.3.** *Let $F: D \to \mathrm{DOM}$ be a continuous functor and let $G$ be a parametrization over $F$. Then $\Sigma(F, G): D \to \mathrm{DOM}$ is a continuous functor.*

**Proof.** For ease of notation we denote $\Sigma(F, G)$ by $H$. As already observed, $H(x)$ is a domain for each $x \in D$. Suppose $x \sqsubseteq y$ and $(u, v) \in H(x)$. Then

$$\begin{aligned} &H^-_{xy}H^+_{xy}(u,v) \\ &\quad = H^-_{xy}(u^{(y)}, v^{(y,u^{(y)})}) = (u^{(y)}{}_{(x)}, v^{(y,u^{(y)})}{}_{(x,u^{(y)}{}_{(x)})}) = (u, v^{(y,u^{(y)})}{}_{(x,u)}) = (u, v). \end{aligned}$$

By a similar calculation, $H^+_{xy}H^-_{xy}(z, w) \sqsubseteq (z, w)$.

Thus $(H^+_{xy}, H^-_{xy})$ is a projection pair in case each component is continuous. We verify this for $H^+_{xy}$, the other case being similar. To show monotonicity, let $(u, v) \sqsubseteq (z, w)$ in $H(x) = \Sigma(F(x), (y)G(x, y))$. Thus $u \sqsubseteq_{F(x)} z$, $v \in G(x, u)$, $w \in G(x, z)$ and $v^{(x,z)} \sqsubseteq w$. First of all, $u^{(y)} \sqsubseteq z^{(y)}$ since $F^+_{xy}$ is monotone. Then

$$v^{(y,u^{(y)})(y,z^{(y)})} = v^{(y,z^{(y)})} = v^{(x,z)(y,z^{(y)})} \sqsubseteq w^{(y,z^{(y)})}.$$

This shows that $H^+_{xy}$ is monotone.

To show continuity, let $(u, v) \in H(x)$ and suppose $(c, d) \in \mathrm{approx}_{H(y)}(u, v)^{(y)}$. We shall find $(a, b) \in \mathrm{approx}_{H(x)}(u, v)$ such that $(c, d) \sqsubseteq (a, b)^{(y)}$. First, by continuity, choose $b_1 \in \mathrm{approx}_{G(x,u)}(v)$ such that $d^{(y,u^{(y)})} \sqsubseteq b_1{}^{(y,u^{(y)})}$.

Then choose $a_1 \in \mathrm{approx}_{F(x)}(u)$ such that $c \sqsubseteq a_1{}^{(y)}$. By the continuity of the functor $(s)G(x, s)$ there is $a_2 \in \mathrm{approx}_{F(x)}(u)$ and $b_2 \in G(x, a_2)_c$ such that $b_1 = b_2{}^{(x,u)}$.

Let $a = a_1 \sqcup a_2$, and let $b = b_2{}^{(x,a)}$. Then $(a, b) \in H(x)_c$. Clearly $a \sqsubseteq u$ and

$$b^{(x,u)} = b_2{}^{(x,u)} = b_1 \sqsubseteq v.$$

Thus $(a, b) \in \mathrm{approx}_{H(x)}(u, v)$, and it remains to show $(c, d) \sqsubseteq (a, b)^{(y)}$. First of all, $c \sqsubseteq a_1{}^{(y)} \sqsubseteq a^{(y)}$. To prove the desired inequality it thus remains to show

$$d^{(y,a^{(y)})} \sqsubseteq b^{(y,a^{(y)})}.$$

Since $a \sqsubseteq u$, this follows from

$$d^{(y,u^{(y)})} \sqsubseteq b_1{}^{(y,u^{(y)})} = b_2{}^{(x,u)(y,u^{(y)})} = b^{(x,u)(y,u^{(y)})} = b^{(y,u^{(y)})}.$$

We conclude that $H^+_{xy}$ is continuous. A similar argument shows that $H^-_{xy}$ is continuous, and hence $(H^+_{xy}, H^-_{xy})$ is a projection pair. That $H$ is a functor is now easily verified.

It remains to show that $H$ is continuous. Suppose $(b, d) \in H(x)_c$. We shall show the existence of $a \in \mathrm{approx}(x)$ and $(\bar{b}, \bar{d}) \in H(a)_c$ such that $(\bar{b}, \bar{d})^{(x)} = (b, d)$. Since $F$ is continuous there is $a' \in \mathrm{approx}(x)$ and $b' \in F(a)_c$ such that $b'^{(x)} = b$. By the continuity of $G$ there is $(a'', b'') \in \mathrm{approx}_{\Sigma(D,F)}(x, b)$ and $d'' \in G(a'', b'')_c$ such that $d''^{(x,b)} = d$. Let $a = a' \sqcup a'' \in \mathrm{approx}(x)$ let $\bar{b} = b'^{(a)}$ and let $\bar{d} = d''^{(a,b)}$. It is easily verified that the embeddings are well-defined and that $(\bar{b}, \bar{d}) \in H(a)_c$. Recall that

$$(\bar{b}, \bar{d})^{(x)} = (\bar{b}^{(x)}, \bar{d}^{(x, \bar{b}^{(x)})}).$$

To show that $(\bar{b}, \bar{d})^{(x)} = (b, d)$ first note that

$$\bar{b}^{(x)} = b'^{(a)(x)} = b'^{(x)} = b.$$

Furthermore, $\bar{d}^{(x, \bar{b}^{(x)})} = d''^{(a, \bar{b})(x,b)} = d''^{(x,b)} = d$. □

Suppose again that $D$ is a domain, $F: D \to \mathrm{DOM}$ is a parametrization and that $G$ is a parametrization over $F$. Then using Theorem 2.1 we may form the domain $\Sigma(\Sigma(D, F), G)$. By Theorem 2.3 we may also consider the functor $\Sigma(F, G): D \to \mathrm{DOM}$ and form the domain $\Sigma(D, \Sigma(F, G))$. The next proposition asserts that these domains are isomorphic so we may identify them. Essentially, they contain dependent ordered triples from $D$, $F$ and $G$.

**Proposition 2.4.** *Let $D$ be a domain and suppose $F$ is a parametrization over $D$ and $G$ is a parametrization over $F$. Then $\Sigma(\Sigma(D, F), G) \cong \Sigma(D, \Sigma(F, G))$.*

**Proof.** Define $\phi: \Sigma(\Sigma(D, F), G) \to \Sigma(D, \Sigma(F, G))$ by $\phi((x, y), z) = (x, (y, z))$. Clearly $\phi$ is a well-defined bijection. It is also a bijection between the compact elements so that all we need to show is that $\phi$ is order-preserving. Working in $\Sigma(\Sigma(D, F), G)$ we have

$$\begin{aligned} ((x, y), z) \sqsubseteq ((u, v), w) &\iff (x, y) \sqsubseteq (u, v) \quad \text{and} \quad z^{(u,v)} \sqsubseteq w \\ &\iff x \sqsubseteq u \quad \text{and} \quad y^{(u)} \sqsubseteq v \quad \text{and} \quad z^{(u,v)} \sqsubseteq w. \end{aligned}$$

Working in $\Sigma(D, \Sigma(F, G))$, and denoting the functor $\Sigma(F, G)$ by $H$, we have

$$\begin{aligned} (x, (y, z)) \sqsubseteq (u, (v, w)) &\iff x \sqsubseteq u \quad \text{and} \quad H^{+}_{xu}(y, z) \sqsubseteq (v, w) \\ &\iff x \sqsubseteq u \quad \text{and} \quad (y^{(u)}, z^{(u, y^{(u)})}) \sqsubseteq (v, w) \\ &\iff x \sqsubseteq u \quad \text{and} \quad y^{(u)} \sqsubseteq v \quad \text{and} \quad z^{(u, y^{(u)})(u,v)} \sqsubseteq w \\ &\iff x \sqsubseteq u \quad \text{and} \quad y^{(u)} \sqsubseteq v \quad \text{and} \quad z^{(u,v)} \sqsubseteq w. \end{aligned}$$

This shows that $\phi$ is order-preserving. □

Suppose $F_1$ is a domain, $F_2$ is a parametrization over $F_1, \ldots,$ and $F_n$ is a parametrization over $F_{n-1}$. Then we shall sometimes use an infix notation, $F_1 \Sigma F_2 \Sigma \cdots \Sigma F_n$ for $\Sigma(\ldots(\Sigma(\Sigma(F_1, F_2), F_3), \ldots, F_n)$ and simply write its elements as $(x_1, \ldots, x_n)$.

Terms in type theory, i.e. expressions for elements in types, will be interpreted as continuous functions, whereas types will be interpreted as parametrizations. Therefore we must make precise what we mean by a function being continuous over a parametrization.

**Definition 2.5.** Let $F: D \to \mathrm{DOM}$ be a continuous functor and let $f: D \to \bigcup\{F(x): x \in D\}$ satisfy $f(x) \in F(x)$ for each $x \in D$. Then $f$ is *p-monotone* if $F^+_{xy}(f(x)) \sqsubseteq f(y)$ whenever $x \sqsubseteq y$, and $f$ is *p-continuous* if, in addition, $f(\bigsqcup A) = \bigsqcup\{F^+[x, \bigsqcup A](f(x)): x \in A\}$ for each directed set $A \subseteq D$.

Of course, the notion of p-continuity only makes sense with respect to a given parametrization, which always will be clear from the context. There is a characterization of p-continuity analogous to Lemma 1.1.

**Lemma 2.6.** *Let $F: D \to \mathrm{DOM}$ be a parametrization and let $f: D \to \bigcup\{F(x): x \in D\}$ satisfy $f(x) \in F(x)$ for each $x \in D$. Then $f$ is p-continuous if and only if $f$ is p-monotone and if $b \in \mathrm{approx}_{F(x)}(f(x))$ then there is $a \in \mathrm{approx}(x)$ such that $b \sqsubseteq f(a)^{(x)}$.*

**Proof.** Suppose first that $f$ is p-continuous and that $b \in \mathrm{approx}_{F(x)}(f(x))$. Then $b \sqsubseteq f(x) = \bigsqcup\{f(a)^{(x)}: a \in \mathrm{approx}(x)\}$. The latter set is directed, so for some $a \in \mathrm{approx}(x)$, $b \sqsubseteq f(a)^{(x)}$.

Conversely, suppose the condition holds. Suppose $A \subseteq D$ is directed and let $x = \bigsqcup A$. Let $b \in \mathrm{approx}_{F(x)}(f(x))$ and choose $a \in \mathrm{approx}(x)$ such that $b \sqsubseteq f(a)^{(x)}$. Since $a$ is compact, there is $y \in A$ such that $a \sqsubseteq y$. It follows by p-monotonicity that $b \sqsubseteq f(a)^{(x)} \sqsubseteq f(y)^{(x)}$ and hence that $f(x) = \bigsqcup\{f(y)^{(x)}: y \in A\}$. □

Next we consider the continuity of the projection functions for the iterated $\Sigma$ construction. Suppose $F_1$ is a domain, $F_2$ is a parametrization over $F_1, \ldots,$ and $F_n$ is a parametrization over $F_{n-1}$. Let $D_i = F_1 \Sigma F_2 \Sigma \cdots \Sigma F_i$ for each $i$, $1 \leq i \leq n$. Define $\bar{F}_i: D_n \to \mathrm{DOM}$ by

$$\bar{F}_i(x_1, \ldots, x_n) = F_i(x_1, \ldots, x_{i-1})$$

and

$$\bar{F}_i^+[(x_1, \ldots, x_n), (y_1, \ldots, y_n)] = F_i^+[(x_1, \ldots, x_{i-1}), (y_1, \ldots, y_{i-1})].$$

It is easily verified that $\bar{F}_i$ is a continuous functor. Now define the projection functions $\pi_i^n: D_n \to \bigcup\{\bar{F}_i(w): w \in D_n\}$ by $\pi_i^n(x_1, \ldots, x_n) = x_i$.

**Lemma 2.7.** *The projection functions $\pi_i^n$ are p-continuous with respect to $\bar{F}_i$.*

**Proof.** Clearly $\pi_i^n$ is p-monotone. Let $b \in \mathrm{approx}(\pi_i^n(x_1, \ldots, x_n))$. Then $b \in \mathrm{approx}_{F_i(x_1,\ldots,x_{i-1})}(x_i)$, and in particular $b \in F_i(x_1, \ldots, x_{i-1})_c$. Thus there is $(a_1, \ldots, a_{i-1}) \in (D_{i-1})_c$ and $b' \in F_i(a_1, \ldots, a_{i-1})_c$ such that

$$F_i^+[(a_1, \ldots, a_{i-1}), (x_1, \ldots, x_{i-1})](b') = b.$$

But then

$$\begin{aligned} b &= F_i^+[(a_1, \ldots, a_{i-1}), (x_1, \ldots, x_{i-1})](b') \\ &= \bar{F}_i^+[(a_1, \ldots, a_{i-1}, b', \perp, \ldots, \perp), (x_1, \ldots, x_n)] \\ &\quad (\pi_i^n(a_1, \ldots, a_{i-1}, b', \perp, \ldots, \perp)) \end{aligned}$$

where $\perp$ is chosen in the appropriate domain. □

Let $F: D \to \mathrm{DOM}$ be a parametrization and consider the embeddings $F_{xy}^+$. For fixed $x$ and $y$, $F_{xy}^+$ is continuous. However we need to be able to say more, namely that $F_{xy}^+(z)$ is continuous in all the variables $x$, $y$ and $z$. To make this precise we first need to restrict ourselves to $x$ and $y$ such that $x \sqsubseteq y$. The proof of the following propositions, needed for the interpretation of $I$-elimination, are left to the reader.

**Proposition 2.8.** *Let $D$ be a domain and let $E = \{(x, y): x, y \in D, x \sqsubseteq y\}$ be ordered coordinate-wise. Then $E$ is a domain and $E_c = \{(a, b) \in E: a, b \in D_c\}$.*

Let $F: D \to \mathrm{DOM}$ be a parametrization and let $G: E \to \mathrm{DOM}$ be defined by $G(x, y) = F(x)$, where $E$ is the domain of Proposition 2.8. By Proposition 1.13, $G$ is a continuous functor. Define $H: \Sigma(E, G) \to \mathrm{DOM}$ by $H(x, y, z) = F(y)$. Then, again using Proposition 1.13 along with Lemma 2.7, $H$ is a continuous functor.

**Proposition 2.9.** *Let $f: \Sigma(E, G) \to \bigcup \{H(w): w \in \Sigma(E, G)\}$ be defined by $f(x, y, z) = F_{xy}^+(z)$. Then $f$ is p-continuous with respect to $H$.*

Suppose $f: D_c \to \bigcup \{F(a): a \in D_c\}$ is such that $f(a) \in F(a)$ for each $a \in D_c$ and $f$ is p-monotone, i.e. $a \sqsubseteq b \Rightarrow f(a)^{(b)} \sqsubseteq f(b)$. Then $f$ has a unique p-continuous extension $\bar{f}$, namely $\bar{f}(x) = \bigsqcup \{f(a)^{(x)}: a \in \mathrm{approx}(x)\}$.

Next we consider the cartesian product of a continuous family of domains. Let $F: D \to \mathrm{DOM}$ be a continuous functor. Define

$$\Pi(D, F) = \{f : f \text{ is p-continuous w.r.t. } F\}$$

and order $\Pi(D, F)$ point-wise, i.e.

$$f \sqsubseteq g \quad \Leftrightarrow \quad f(x) \sqsubseteq_{F(x)} g(x) \quad \text{for each } x \in D.$$

Then $\Pi(D, F)$ with the point-wise ordering is the *cartesian product of $F$ over $D$*.

**Theorem 2.10.** *Let $F: D \to \mathrm{DOM}$ be a continuous functor. Then $\Pi(D, F)$ is a domain.*

**Proof.** For ease of notation we write $\Pi$ for $\Pi(D, F)$. Clearly, $\Pi$ is partially ordered with a least element. Let $A \subseteq \Pi$ be directed. We must show $\bigsqcup A \in \Pi$.

For each $x \in D$, the set $A_x = \{f(x) : f \in A\}$ is directed in $F(x)$ so we define

$$k(x) = \bigsqcup A_x \in F(x).$$

The function $k$ will be our sought supremum. To show that $k$ is p-monotone let $x \sqsubseteq y$ in $D$. Then $f(x)^{(y)} \sqsubseteq f(y)$ for each $f \in A$ and hence

$$\begin{aligned} k(x)^{(y)} &= (\bigsqcup \{f(x) : f \in A\})^{(y)} \\ &= \bigsqcup \{f(x)^{(y)} : f \in A\} \sqsubseteq \bigsqcup \{f(y) : f \in A\} = k(y). \end{aligned}$$

To show p-continuity, let $b \in \mathrm{approx}_{F(x)}(k(x))$. Then $b \sqsubseteq f(x)$ for some $f \in A$, since $b$ is compact. By the p-continuity of $f$ there is $a \in \mathrm{approx}(x)$ such that $b \sqsubseteq f(a)^{(x)}$. But then $b \sqsubseteq k(a)^{(x)}$. The verification that $k = \bigsqcup A$ is immediate, so $\Pi$ is a cpo.

Let $a \in D_c$ and $b \in F(a)_c$ and define

$$\begin{aligned} \langle a; b\rangle(x) &= b^{(x)} \quad &&\text{if } a \sqsubseteq x, \\ &= \bot_{F(x)} &&\text{else.} \end{aligned}$$

It is routinely verified that $\langle a; b\rangle \in \Pi$ and that $\langle a; b\rangle$ is compact. We say that $\langle a; b\rangle$ is a *basic* compact element. Of course, $\langle a; b\rangle$ is the least function in $\Pi$ sending $a$ to $b$. Let $B = \{\langle a_i; b_i\rangle ; a_i \in D_c,\ b_i \in F(a_i)_c,\ i = 1, \ldots, n\}$ be consistent in $\Pi$, say bounded by $g$. For each $x \in D$ let $I_x = \{i : a_i \sqsubseteq x\}$. Then the set $\{b_i^{(x)} : i \in I_x\} \subseteq F(x)$ is bounded by $g(x)$ and hence $h(x) = \bigsqcup \{b_i^{(x)} : i \in I_x\} \in F(x)_c$ is well-defined ($= \bot_{F(x)}$ in case $I_x = \emptyset$). It is easily verified that $h$ is p-continuous and that $h = \bigsqcup B$. To show that $h \in \Pi_c$ suppose $h \sqsubseteq \bigsqcup A$ where $A \subseteq \Pi$ is directed. Then for each $i$, $h(a_i) \sqsubseteq \bigsqcup \{f(a_i) : f \in A\}$ and hence there is $f_i \in A$ such that $h(a_i) \sqsubseteq f_i(a_i)$. Choose $f \in A$ such that $f_i \sqsubseteq f$ for $i = 1, \ldots, n$. Then it is clear that $h \sqsubseteq f$.

Let $k \in \Pi$ and let $A = \{\langle a; b\rangle : a \in D_c, b \in F(a)_c,$ and $\langle a; b\rangle \sqsubseteq k\}$. Then $\bigsqcup A(x) = \bigsqcup \{\langle a; b\rangle(x) : \langle a; b\rangle \in A\}$ exists for each $x$ by consistent completeness. We show that $k(x) = \bigsqcup A(x)$ for each $x$ and hence that $\bigsqcup A \in \Pi$. In the non-trivial direction, let $b' \in \mathrm{approx}_{F(x)}(k(x))$. We shall find $\langle a; b\rangle \in A$ such that $\langle a; b\rangle(x) = b'$, i.e. $b^{(x)} = b'$. By the p-continuity of $k$ there is $a' \in \mathrm{approx}(x)$ such that $b' \sqsubseteq k(a')^{(x)}$. By the continuity of $F$ there is $a'' \in \mathrm{approx}(x)$ and $b'' \in F(a'')_c$ such that $b''^{(x)} = b'$. Let $a = a' \sqcup a''$ and $b = b''^{(a)}$. Then $\langle a; b\rangle \in A$ and $\langle a; b\rangle(x) = b'$. This proves our equality.

Let $A'$ be the closure of $A$ under finite consistent suprema. Then $A'$ is a directed set and $k = \bigsqcup A'$ in $\Pi$. In case $k$ is compact then $k$ must equal some element in $A'$. It follows that the compact element in $\Pi$ are precisely the suprema of finite consistent sets of basic compact functions of the form $\langle a; b\rangle$.

The above argument also shows that $\Pi$ is algebraic and consistently complete. □

In the special case that $F : D \to \mathbf{DOM}$ is the constant parametrization of Example 1.10, i.e. $F(x) = E$ for $x \in D$, then $\Pi(D, F) = D \to E$.

Analogous to the case of disjoint sums we must also consider the cartesian product as a functor. Let $F: D \to \mathrm{DOM}$ be a continuous functor and let $G$ be a parametrization over $F$. Define a functor $\Pi(F, G): D \to \mathrm{DOM}$ by

$$\Pi(F, G)(x) = \Pi(F(x), (s)G(x, s))$$

and, when $x \sqsubseteq y$,

$$\Pi(F, G)^{+}_{xy}(f) = (s)G^{+}[(x, F^{-}_{xy}(s)), (y, s)]fF^{-}_{xy}(s) = (s)f(s_{(x)})^{(y,s)}$$

and

$$\Pi(F, G)^{-}_{xy}(g) = (t)G^{-}[(x, t), (y, F^{+}_{xy}(t))]gF^{+}_{xy}(t) = (t)g(t^{(y)})_{(x,t)}.$$

These definitions become clear when considering the following diagram:

$$\begin{tikzcd}
F(y) \arrow[rr, "g"] \arrow[d, shift right, "F^{-}_{xy}"'] & & \bigcup G(y, s) \arrow[d, shift right, "{G^{-}[(x, t), (y, s)]}"'] \\
F(x) \arrow[rr, "f"'] \arrow[u, shift right, "F^{+}_{xy}"'] & & \bigcup G(x, t) \arrow[u, shift right, "{G^{+}[(x, t), (y, s)]}"']
\end{tikzcd}$$

**Theorem 2.11.** *Let $F: D \to \mathrm{DOM}$ be a continuous functor and let $G$ be a parametrization over $F$. Then $\Pi(F, G): D \to \mathrm{DOM}$ is a continuous functor.*

**Proof.** For ease of notation we denote $\Pi(F, G)$ by $H$. By Proposition 1.13 and Theorem 2.10, $H(x)$ is a domain for each $x \in D$. Suppose $x \sqsubseteq y$ and $f \in \Pi(F(x), (t)G(x, t))$. Then

$$\begin{aligned} H^{-}_{xy}H^{+}_{xy}(f) &= (t)((s)f(s_{(x)})^{(y,s)})(t^{(y)})_{(x,t)} \\ &= (t)f(t^{(y)}{}_{(x)})^{(y,t^{(y)})}{}_{(x,t)} = (t)f(t)^{(y,t^{(y)})}{}_{(x,t)} = (t)f(t) = f. \end{aligned}$$

On the other hand, if $g \in \Pi(F(y), (s)G(y, s))$ then

$$\begin{aligned} H^{+}_{xy}H^{-}_{xy}(g) &= (s)((t)g(t^{(y)})_{(x,t)})(s_{(x)})^{(y,s)} \\ &= (s)g(s_{(x)}{}^{(y)})_{(x,s_{(x)})}{}^{(y,s)} \sqsubseteq (s)g(s_{(x)}{}^{(y)})^{(y,s)} \sqsubseteq (s)g(s) = g. \end{aligned}$$

The first inequality is obtained from the morphisms being projection pairs, while the second is just p-monotonicity for $g$.

Trivially, $H^{+}_{xy}$ and $H^{-}_{xy}$ are monotone. To see that $H^{+}_{xy}$ is continuous, let $A \subseteq H(x)$ be directed. Then

$$\begin{aligned} H^{+}_{xy}(\bigsqcup A) &= (s)(\bigsqcup A)(s_{(x)})^{(y,s)} \\ &= (s)(\bigsqcup \{f(s_{(x)}) : f \in A\})^{(y,s)} = \bigsqcup \{(s)f(s_{(x)})^{(y,s)} : f \in A\} \\ &= \bigsqcup H^{+}_{xy}(A). \end{aligned}$$

Similarly, $H^{-}_{xy}$ is continuous. Thus $(H^{+}_{xy}, H^{-}_{xy})$ is a projection pair. It is easily verified that the morphisms commute properly so that $H$ is a functor.

It remains to show that $H$ is continuous. Let $\langle b; d\rangle \in H(x)_{c}$ be a basic compact function defined in the proof of Theorem 2.10, where $b \in F(x)_{c}$ and $d \in G(x, b)_{c}$.

By the continuity of $F$ there is $a' \in \mathrm{approx}(x)$ and $b' \in F(a')_c$ such that $b'^{(x)} = b$. By the continuity of $G$ there is $a'' \in \mathrm{approx}(x)$, $b'' \in F(a'')_c$ such that $b''^{(x)} \sqsubseteq b$, and $d'' \in G(a'', b'')_c$ such that $d''^{(x,b)} = d$. Let $a = a' \sqcup a''$, $\bar{b} = b'^{(a)}$, and let $\bar{d} = d''^{(a,\bar{b})}$. Clearly $\bar{d}$ is defined, $\bar{b} \in F(a)_c$ and $\bar{d} \in G(a, \bar{b})_c$, so $\langle \bar{b}; \bar{d} \rangle \in H(a)_c$. Moreover $\bar{b}^{(x)} = b$ and $\bar{d}^{(x,b)} = d$.

We claim that $\langle \bar{b}; \bar{d} \rangle^{(x)} = \langle b; d \rangle$. Given $z \in F(x)$, then by the monotonicity of the morphisms, $b \sqsubseteq z \Leftrightarrow \bar{b} \sqsubseteq z_{(a)}$. So suppose $b \sqsubseteq z$. Then

$$\begin{aligned}\langle \bar{b}; \bar{d} \rangle^{(x)}(z) &= \langle \bar{b}; \bar{d} \rangle(z_{(a)})^{(x,z)} = \bar{d}^{(a, z_{(a)})(x,z)} \\ &= \bar{d}^{(x,z)} = \bar{d}^{(x,b)(x,z)} = d^{(x,z)} = \langle b; d \rangle(z).\end{aligned}$$

A similar calculation works also in the case $\neg(b \sqsubseteq z)$, recalling that embeddings are strict, i.e. send $\perp$ to $\perp$. This proves that each basic compact function $\langle b; d \rangle \in H(x)_c$ is the image of a basic compact function $\langle \bar{b}; \bar{d} \rangle \in H(a)_c$ for some $a \in \mathrm{approx}(x)$. This suffices to prove the continuity of $H$ since embeddings preserve arbitrary suprema. □

We shall now consider substitution into p-continuous functions. First we show that substituting a continuous function into a p-continuous function preserves p-continuity.

**Lemma 2.12.** *Let $D$ and $E$ be domains, $f: D \to E$ continuous, $F: E \to \mathrm{DOM}$ a parametrization, and let $g \in \Pi(E, F)$. Let $G: D \to \mathrm{DOM}$ be defined by $G = F \circ f$ and define $h: D \to \bigcup \{G(x): x \in D\}$ by $h(x) = g(f(x))$. Then $h \in \Pi(D, G)$.*

**Proof.** Clearly $h$ is p-monotone. To prove p-continuity suppose $d \in \mathrm{approx}_{G(x)}(h(x))$. By the p-continuity of $g$ choose $b \in \mathrm{approx}_E(f(x))$ such that $d \sqsubseteq g(b)^{(f(x))}$. By the continuity of $f$ choose $a \in \mathrm{approx}_D(x)$ such that $b \sqsubseteq f(a)$. Then

$$d \sqsubseteq g(b)^{(f(x))} = g(b)^{(f(a))(f(x))} \sqsubseteq g(f(a))^{(f(x))} = G^{+}_{ax}(h(a)). \quad \square$$

Now consider substitution with regard to disjoint sums. Clearly, in order not to destroy dependencies, it only makes sense to substitute in the rightmost coordinate. Let $D$ be a domain and $F: D \to \mathrm{DOM}$ a continuous functor. Let $f: D \to \bigcup \{F(x): x \in D\}$ be such that $f(x) \in F(x)$ for each $x \in D$ and define $f^*: D \to \Sigma(D, F)$ by $f^*(x) = (x, f(x))$. The function $f^*$ is the *section* of $\Sigma(D, F)$ determined by $f$.

**Lemma 2.13.** *The section $f^*$ is continuous if and only if $f \in \Pi(D, F)$.*

**Proof.** First assume $f \in \Pi(D, F)$. Then clearly $f^*$ is monotone. Suppose $(b, d) \in \mathrm{approx}(f^*(x))$, i.e. $b \in \mathrm{approx}(x)$, $d \in F(b)_c$, and $d^{(x)} \sqsubseteq f(x)$. By the p-continuity of $f$, there is $a' \in \mathrm{approx}(x)$ such that $d^{(x)} \sqsubseteq f(a')^{(x)}$. Let $a = a' \sqcup b$. Then

$$d^{(a)(x)} = d^{(x)} \sqsubseteq f(a')^{(x)} = f(a')^{(a)(x)}.$$

Thus $d^{(a)} \sqsubseteq f(a')^{(a)} \sqsubseteq f(a)$, i.e. $(b, d) \sqsubseteq f^*(a)$.

Conversely, suppose $f^*$ is continuous. If $x \sqsubseteq y$ then $f^*(x) \sqsubseteq f^*(y)$, i.e. $(x, f(x)) \sqsubseteq (y, f(y))$. This means that $f(x)^{(y)} \sqsubseteq f(y)$ so $f$ is p-monotone. Referring to the proof of Theorem 2.1 we also have

$$\begin{aligned}(x, f(x)) = f^*(x) &= \bigsqcup \{(a, f(a)) : a \in \mathrm{approx}(x)\} \\ &= (x, \bigsqcup \{f(a)^{(x)} : a \in \mathrm{approx}(x)\}).\end{aligned}$$

Thus $f \in \Pi(D, F)$. □

**Lemma 2.14.** *Let $F : D \to \mathrm{DOM}$ be a continuous functor, $f \in \Pi(D, F)$, and $G$ a parametrization over $F$. Define $H : D \to \mathrm{DOM}$ by $H(x) = G(x, f(x))$ and, when $x \sqsubseteq y$, $H^+_{xy} = G^+[(x, f(x)), (y, f(y))]$, and $H^-_{xy} = G^-[(x, f(x)), (y, f(y))]$. Then $H$ is a continuous functor.*

**Proof.** The section $f^*$ of $f$ is continuous by Lemma 2.13 and hence $H$ is continuous by Proposition 1.13. □

Similarly we can substitute in the rightmost coordinate of a p-continuous function. Let $F : D \to \mathrm{DOM}$ be a continuous functor, $f \in \Pi(D, F)$, and $G$ a parametrization over $F$, and let $H$ be the functor defined in Lemma 2.14.

**Lemma 2.15.** *Let $g \in \Pi(\Sigma(D, F), G)$ and define $h : D \to \bigcup \{H(x) : x \in D\}$ by $h(x) = g(x, f(x))$. Then $h \in \Pi(D, H)$.*

**Proof.** Follows from Lemma 2.12 and Lemma 2.13. □

Of course, we may iterate Lemmata 2.13 to 2.15. For example, suppose $f \in \Pi(D, F)$ and $g \in \Pi(\Sigma(D, F), G)$. Define $h : D \to D\, \Sigma\, F\, \Sigma\, G$ by $h(x) = (x, f(x), g(x, f(x))) = g^* f^*(x)$. Then $h$ is continuous since it is the composition of two continuous functions by Lemma 2.13. Similarly, let $K$ be a parametrization over $G$ and define $L : D \to \mathrm{DOM}$ by $L(x) = K(x, f(x), g(x, f(x))) = K \circ g^* f^*(x)$. Then $L$ is a parametrization by the argument of Lemma 2.14. Thus we have by induction:

**Lemma 2.16.** *Let $D$ be a domain, $F_1$ a parametrization over $D$, $F_2$ a parametrization over $F_1, \ldots,$ and $F_n$ a parametrization over $F_{n-1}$. Let $f_1 \in \Pi(D, F_1)$, $f_2 \in \Pi(D\, \Sigma\, F_1, F_2), \ldots,$ and $f_n \in \Pi(D\, \Sigma\, F_1\, \Sigma \cdots \Sigma\, F_{n-1}, F_n)$.*

(i) *Define* $g: D \to D \,\Sigma\, F_1 \,\Sigma \cdots \Sigma\, F_n$ *by*

$$g(x) = (x, f_1(x), f_2(x, f_1(x)), \ldots, f_n(x, f_1(x), \ldots, f_{n-1}(x, f_1(x), \ldots)))$$
$$= f_n^* \cdots f_2^* f_1^*(x).$$

*Then* $g$ *is continuous.*

(ii) *Let* $G$ *be a parametrization over* $F_n$ *and define* $H: D \to \mathrm{DOM}$ *by* $H = G \circ g$. *Then* $H$ *is a parametrization.*

(iii) *Let* $k \in \Pi(D \,\Sigma\, F_1 \,\Sigma \cdots \Sigma\, F_n, G)$ *and define* $h: D \to \bigcup \{H(x) : x \in D\}$ *by*

$$h(x) = k(x, f_1(x), f_2(x, f_1(x)), \ldots, f_n(x, f_1(x), \ldots, f_{n-1}(x, f_1(x), \ldots)))$$
$$= k f_n^* \cdots f_2^* f_1^*(x).$$

*Then* $h \in \Pi(D, H)$.

Let $D$ and $E$ be domains and let $F: D \times E \to \mathrm{DOM}$ be a continuous functor. Then fixing one argument, say $x \in D$, gives us a continuous functor $(y)F(x, y): E \to \mathrm{DOM}$ by Proposition 1.13. Suppose $f: D \times E \to \bigcup \{F(x, y) : x \in D, y \in E\}$. Again fixing one argument, say $x \in D$, we say that $f$ is p-continuous in the other coordinate if $(y)f(x, y)$ is p-continuous w.r.t. $(y)F(x, y)$ for each choice of $x$.

**Lemma 2.17.** *Let* $D$ *and* $E$ *be domains and assume* $F: D \times E \to \mathrm{DOM}$ *is a continuous functor and* $f: D \times E \to \bigcup \{F(x, y) : x \in D, y \in E\}$. *Then* $f \in \Pi(D \times E, F)$ *if and only if* $f$ *is p-continuous in each argument.*

**Proof.** Assume $f$ is p-continuous in each coordinate. We prove $f \in \Pi(D \times E, F)$, the other direction being even simpler. Using Lemma 2.13 and the notation there, it suffices to show that $f^*$ is continuous. Fix $y \in E$. By Lemma 2.13 and the hypothesis, $((x)f(x, y))^*$ is continuous. Define $h: \Sigma(D, (x)F(x, y)) \to \Sigma(D \times E, F)$ by $h(x, z) = (x, y, z)$ . Then clearly $h$ is continuous. But $(x)f^*(x, y) = h \circ ((x)f(x, y))^*$ so $(x)f^*(x, y)$ is continuous. Similarly, $(y)f^*(x, y)$ is continuous for each $x \in D$. Thus $f^*$ is continuous since continuous in each coordinate. □

Here is a lemma about forming pairs for the $\Sigma$ functor, which will be used when interpreting the disjoint sum.

**Lemma 2.18.** *Let* $F: D \to \mathrm{DOM}$ *be a parametrization and let* $G$ *be a parametrization over* $F$. *Let* $f \in \Pi(D, F)$ *and let* $g \in \Pi(D, (w)G(w, f(w)))$. *Define* $h: D \to \bigcup \{\Sigma(F, G)(w) : w \in D\}$ *by* $h(w) = (f(w), g(w))$. *Then* $h \in \Pi(D, \Sigma(F, G))$.

**Proof.** Again we use Lemma 2.13. First observe that $(w)G(w, f(w)) = G \circ f^*$. Define $f': \Sigma(D, G \circ f^*) \to \Sigma(D, \Sigma(F, G))$ by $f'(w, v) = (w, (f(w), v))$. We omit the routine verification that $f'$ is continuous. Note that for $w \in D$,

$$f' \circ g^*(w) = f'(w, g(w)) = (w, (f(w), g(w))).$$

Thus $h^* = f' \circ g^*$ and hence $h^*$ is continuous, so $h \in \Pi(D, \Sigma(F, G))$ by Lemma 2.13. $\square$

We must also consider a generalization of the initial segment of a domain construction of Example 1.11, in order to interpret the $I$-type of intuitionistic type theory. Let $F: D \to \mathrm{DOM}$ be a parametrization and define $I(F): \Sigma(D, F) \to \mathrm{DOM}$ by

$$I(F)(x, y) = F(x)^y$$

and, when $(x, y) \sqsubseteq (u, v)$, define $I(F)^+[(x, y), (u, v)] : F(x)^y \to F(u)^v$ by

$$I(F)^+[(x, y), (u, v)](z) = F^+_{xu}(z).$$

**Lemma 2.19.** *Let $F: D \to \mathrm{DOM}$ be a parametrization. Then $I(F): \Sigma(D, F) \to \mathrm{DOM}$ is a parametrization.*

**Proof.** Clearly $I(F)(x, y)$ is a domain for each $(x, y) \in \Sigma(D, F)$. Suppose $(x, y) \sqsubseteq (u, v)$, i.e. $x \sqsubseteq u$ and $y^{(u)} \sqsubseteq v$. For $z \in F(x)^y$, $z^{(u)} \sqsubseteq y^{(u)} \sqsubseteq v$, so $F^+_{xu}(z) \in F(u)^v$. In particular, if $a \in (F(x)^y)_c = F(x)_c \cap F(x)^y$ then $F^+_{xu}(a) \in (F(u)^v)_c$. Using Proposition 1.5 one easily verifies that $I(F)^+[(x, y), (u, v)]$ is a projection embedding. The required composition properties obviously holds so that $I(F)$ is a functor. To show the continuity of $I(F)$ suppose $d \in I(F)(x, y)_c$, i.e. $d \in F(x)_c$ and $d \sqsubseteq y$. By the continuity of $F$ choose $a \in \mathrm{approx}(x)$ and $b \in F(a)_c$ such that $b^{(x)} = d$. Then $(a, b) \in \Sigma(D, F)_c$ and $(a, b) \sqsubseteq (x, y)$ since $b^{(x)} = d \sqsubseteq y$. Now, $b \in I(F)(a, b) = F(a)^b$ so $I(F)^+[(a, b), (x, y)](b) = b^{(x)} = d$. $\square$

For $x, y \in D$, we denote the greatest lower bound of $x$ and $y$ in the domain $D$ by $x \sqcap y$. Clearly, $x \sqcap y$ exists, in fact

$$x \sqcap y = \bigsqcup \{a \in D_c : a \sqsubseteq x, a \sqsubseteq y\} = \bigsqcup (\mathrm{approx}(x) \cap \mathrm{approx}(y)).$$

**Lemma 2.20.** *Let $F: D \to \mathrm{DOM}$ be a parametrization and suppose $f, g \in \Pi(D, F)$. Define $h: D \to \bigcup \{F(w) : w \in D\}$ by $h(w) = f(w) \sqcap g(w)$. Then $h \in \Pi(D, F)$.*

**Proof.** Using the notation for sections we have $h^*(w) = f^*(w) \sqcap g^*(w)$. Now, $f^*$ and $g^*$ are continuous by Lemma 2.13 and hence, as is well-known, $h^*$ is continuous. But then $h \in \Pi(D, F)$ again by Lemma 2.13. $\square$

**Theorem 2.21.** *Let $F: D \to \mathrm{DOM}$ be a parametrization and suppose $f, g \in \Pi(D, F)$. Define $I(F, f, g); D \to \mathrm{DOM}$ by $I(F, f, g)(x) = I(F)(x, f(x) \sqcap g(x))$ and, when $x \sqsubseteq y$, $I(F, f, g)^+_{xy} = I(F)^+[(x, f(x) \sqcap g(x)), (y, f(y) \sqcap g(y))]$. Then $I(F, f, g): D \to \mathrm{DOM}$ is a parametrization.*

**Proof.** By Lemma 2.19, 2.20 and 2.14. $\square$

Clearly, if $h \in \Pi(D, I(F, f, g))$ then also $h \in \Pi(D, F)$ when considering $h$ as a function $h: D \to \bigcup \{F(w): w \in D\}$.

Our final concern in this section is the important curry operation. Let $F: D \to \mathrm{DOM}$ be a continuous functor and let $G$ be a parametrization over $F$. Define curry: $\Pi(\Sigma(D, F), G) \to \Pi(D, \Pi(F, G))$ by

$$\mathrm{curry}(f) = (x)(y)f(x, y)$$

where, as usual, $(z)$ denotes abstraction in the variable $z$.

**Theorem 2.22.** *The function* curry *is an isomorphism between* $\Pi(\Sigma(D, F), G)$ *and* $\Pi(D, \Pi(F, G))$.

**Proof.** Throughout the proof let $g \in \Pi(\Sigma(D, F), G)$. First we show that $(y)g(x, y) \in \Pi(F, G)(x) = \Pi(F(x), (s)G(x, s))$ for each fixed $x \in D$. Clearly $(y)g(x, y)$ is p-monotone. To show p-continuity, assume $d \in \mathrm{approx}(g(x, y))$. Then by the p-continuity of $g$ there is $(a, b) \in \mathrm{approx}((x, y))$ such that

$$d \sqsubseteq g(a, b)^{(x,y)} = g(a, b)^{(x, b^{(x)})(x,y)} \sqsubseteq g(x, b^{(x)})^{(x,y)}.$$

This proves p-continuity since $b^{(x)} \in F(x)_c$.

Next we show $\mathrm{curry}(g) \in \Pi(D, \Pi(F, G))$. Again we leave the p-monotonicity for the reader. For p-continuity we need as usual only consider basic compact functions. So assume $\langle b; d\rangle \sqsubseteq \mathrm{curry}(g)(x)$. This is equivalent to $d \sqsubseteq g(x, b)$. We must show the existence of $a \in \mathrm{approx}(x)$ such that

$$\langle b; d\rangle \sqsubseteq \Pi(F, G)^{+}_{ax}(\mathrm{curry}(g)(a)) = \mathrm{curry}(g)(a)^{(x)}.$$

For this it suffices that

$$d \sqsubseteq (((s)g(a, s))^{(x)})(b) = g(a, b_{(a)})^{(x,b)}.$$

By the continuity of $g$, there is $(a', b') \sqsubseteq (x, b)$ such that

$$d \sqsubseteq g(a', b')^{(x,b)}.$$

By the continuity of $F$, there is $a'' \in \mathrm{approx}(x)$ and $b'' \in F(a'')_c$ such that $b''^{(x)} = b$. Let $a = a' \sqcup a''$ and let $\bar{b} = b''^{(a)}$. It is easily verified that $(a', b') \sqsubseteq (a, \bar{b}) \sqsubseteq (x, b)$. Thus

$$d \sqsubseteq g(a', b')^{(x,b)} = g(a', b')^{(a,\bar{b})(x,b)} \sqsubseteq g(a, \bar{b})^{(x,b)} = g(a, b_{(a)})^{(x,b)}.$$

This proves the $\mathrm{curry}(g) \in \Pi(D, \Pi(F, G))$.

To show that curry is continuous it suffices to verify that if $a \in \mathrm{approx}(x)$, $b \in F(a)_c$ and $d \in G(a, b)_c$ then $\langle a; \langle b; d\rangle\rangle = \mathrm{curry}(\langle (a, b); d\rangle)$. Leaving this to the reader along with the verification that curry is order-preserving, it remains to show that curry is onto. Let $f \in \Pi(D, \Pi(F, G))$, and define $g: \Sigma(D, F) \to \bigcup \{G(x, y): (x, y) \in \Sigma(D, F)\}$ by $g(x, y) = f(x)(y)$. We must show that $g$ is p-continuous.

First we show that $g$ is p-monotone. Suppose $(x, y) \sqsubseteq (u, v)$, i.e. $x \sqsubseteq u$ and $y^{(u)} \sqsubseteq v$. We need to show that $f(x)(y)^{(u,v)} \sqsubseteq f(u)(v)$. By the p-monotonicity of $f$, $f(x)^{(u)} \sqsubseteq f(u)$, so in particular

$$f(x)(v_{(x)})^{(u,v)} \sqsubseteq f(u)(v).$$

But $y \sqsubseteq v_{(x)}$ so $f(x)(y)^{(x, v_{(x)})} \sqsubseteq f(x)(v_{(x)})$. So

$$f(x)(y)^{(x,v_{(x)})(u,v)} \sqsubseteq f(x)(v_{(x)})^{(u,v)} \sqsubseteq f(u)(v)$$

i.e. $f(x)(y)^{(u,v)} \sqsubseteq f(u)(v)$.

Finally, to show that $g$ is p-continuous assume that $d \in \text{approx}(g(x, y))$. By the continuity of $G$, let $(a', b') \in \Sigma(D, F)_c$ and $d' \in G(a', b')_c$ such that $d'^{(x,y)} = d$. By the continuity of $f(x)$, let $b'' \in F(x)_c$, $b'' \sqsubseteq y$ such that $d \sqsubseteq f(x)(b'')^{(x,y)}$. Let $\bar{b} = b'^{(x)} \sqcup b'' \in F(x)_c$. Then, since $f(x)$ is p-monotone, $d \sqsubseteq f(x)(\bar{b})^{(x,y)}$. Let $\bar{d} = d'^{(x,\bar{b})}$, so $\bar{d}^{(x,y)} = d$. Consider $\langle \bar{b}, \bar{d} \rangle \in \Pi(F, G)(x)_c$. By the above, $\bar{d} \sqsubseteq f(x)(\bar{b})$ so $\langle \bar{b}, \bar{d} \rangle \sqsubseteq f(x)$ in $\Pi(F, G)(x)$. By the continuity of $f$ there is $a'' \in \text{approx}(x)$ such that $\langle \bar{b}, \bar{d} \rangle \sqsubseteq f(a'')^{(x)}$. By the continuity of $F$ choose $a''' \in \text{approx}(x)$ and $b''' \in F(a''')_c$ such that $b'''^{(x)} = \bar{b}$. Let $a = a' \sqcup a'' \sqcup a'''$ and let $b = b'''^{(a)}$, so $b^{(x)} = \bar{b}$. We claim that $d \sqsubseteq g(a, b)^{(x,y)} = f(a)(b)^{(x,y)}$, which completes the proof. Note that

$$\langle \bar{b}, \bar{d} \rangle \sqsubseteq f(a'')^{(x)} = f(a'')^{(a)(x)} \sqsubseteq f(a)^{(x)} = (s)f(a)(s_{(a)})^{(x,s)},$$

so

$$\langle \bar{b}, \bar{d} \rangle(\bar{b}) = \bar{d} \sqsubseteq f(a)(\bar{b}_{(a)})^{(x,\bar{b})} = f(a)(b)^{(x,\bar{b})}.$$

But then

$$d = \bar{d}^{(x,y)} \sqsubseteq f(a)(b)^{(x,\bar{b})(x,y)} = f(a)(b)^{(x,y)} = g(a, b)^{(x,y)}. \qquad \square$$

To see the usefulness of currying, let us consider a simple example. Suppose that $D$, $F$ and $G$ are as above and let $f \in \Pi(D, \Pi(F, G))$ and $g \in \Pi(D, F)$. Define $\not{f}: D \to \bigcup \{G(w, g(w)) : w \in D\}$ by $\not{f}(w) = f(w)g(w)$. Then

$$\not{f}(w) = \text{curry}^{-1}(f)(w, g(w)),$$

and the latter is p-continuous by Theorem 2.22 and Lemma 2.15.

## 3. Effective parametrizations

There is a well-known theory of effectivity on domains using enumerations. In this section we will briefly review this theory and then extend the theory to parametrizations, i.e. continuous functors over domains. We will show that all our constructs, in particular the disjoint sum and the cartesian product, are effective.

**Definition 3.1.** Let $D = (D; \sqsubseteq, \perp)$ be a domain. Then $D$ is *effective* if the structure $D_c = (D_c; \sqcup, \text{Cons}, \perp)$ is *computable*. The latter means that there is an r.e. set $\Omega_\alpha \subseteq \omega$ and a surjection $\alpha : \Omega_\alpha \to D_c$ such that
(i) the relation $\text{Cons}(a, b) \Leftrightarrow \exists c\,(a \sqsubseteq c \;\&\; b \sqsubseteq c)$ is $\alpha$-decidable,
(ii) the partial function $(a, b) \to a \sqcup b$ is $\alpha$-computable,
(iii) equality on $D_c$ is $\alpha$-decidable, and
(iv) there is a designated $n \in \Omega_\alpha$ such that $\alpha(n) = \perp$.

To say that a relation (function) is $\alpha$-decidable ($\alpha$-computable) means that there is a recursive relation (function) which decides (computes) the corresponding relation (function) on the index set $\Omega_\alpha$. To exemplify, (iii) means that there is a recursive relation $R$ such that for $m, n \in \Omega_\alpha$,

$$\alpha(m) = \alpha(n) \quad \Leftrightarrow \quad R(m, n).$$

When we want to make the chosen numbering $\alpha$ explicit, we write $(D, \alpha)$. It is appropriate to do so since an effective domain may have many inequivalent effective numberings. Note that if $(D, \alpha)$ is an effective domain then the relation $\sqsubseteq$ is $\alpha$-decidable on $D_c$ since

$$a \sqsubseteq b \quad \Leftrightarrow \quad \text{Cons}(a, b) \;\&\; a \sqcup b = b.$$

**Definition 3.2.** Let $(D, \alpha)$ and $(E, \beta)$ be effective domains. Then
(i) $x \in D$ is *effective* if approx$(x)$ is $\alpha$-semidecidable, and
(ii) a continuous function $f : D \to E$ is $(\alpha, \beta)$-*effective* defined by $R(a, b) \Leftrightarrow b \sqsubseteq$ f(a) is $(\alpha, \beta)$-semidecidable.

The effective elements of a domain are those which can be effectively approximated by compact elements. Of course, every compact element is effective. A function is effective if its values on compact elements can be effectively approximated. The following theorem is well-known.

**Theorem 3.3.** *Let $D$ and $E$ be effective domains.*
(i) *If $f : D \to E$ is effective and $x \in D$ is effective then $f(x) \in E$ is effective.*
(ii) *$D \times E$, $D + E$, $D_\perp$ and $D \to E$ are effective domains.*
(iii) *A continuous function $f : D \to E$ is effective if and only if $f$ as an element of $D \to E$ is effective.*

Before considering the effectivity of parametrizations we prove a useful result about the effectivity of projection pairs.

**Proposition 3.4.** *Let $D$ and $E$ be effective domains and suppose $(f^+, f^-)$ is a projection pair from $D$ to $E$. Then $f^+ \restriction D_c : D_c \to E_c$ is computable if and only if $f^+$ and $f^-$ are both effective.*

By $f^+ \upharpoonright D_c$ being computable we mean that there is a recursive *tracking function* for $f^+ \upharpoonright D_c$, i.e. there is a recursive function which given an index for $a \in D_c$ computes an index for $f^+(a) \in E_c$. We always keep this distinction between computable and effective.

For clarity and brevity all our proofs will be informal in the sense that the numberings will be suppressed whenever convenient to do so. Thus we will informally manipulate elements in the considered structure rather than their codes.

**Proof.** Suppose first that $f^+ \upharpoonright D_c$ is computable. Then $f^+$ is effective since $b \sqsubseteq f^+(a)$ is even decidable. Furthermore $a \sqsubseteq f^-(b) \Leftrightarrow f^+(a) \sqsubseteq b$ so $f^-$ is also effective.

Conversely, suppose $f^+$ and $f^-$ are effective. First observe that $f^+(a) = b$ is semidecidable for $a \in D_c$ and $b \in E_c$. For $b \sqsubseteq f^+(a)$ is semidecidable by the effectivity of $f^+$ and $f^+(a) \sqsubseteq b \Leftrightarrow a \sqsubseteq f^-(b)$ is semidecidable by the effectivity of $f^-$. Define $\sigma$ for $a \in D_c$ by

$$\sigma(a) = \text{some } b \; [b \in \text{approx}(f^+(a)) \; \& \; f^+(a) = b].$$

The relation within the brackets is semidecidable so $\sigma$ is partial computable, and $\sigma(a)$ is defined since $f^+(a) \in E_c$. $\square$

**Definition 3.5.** Let $(D, \alpha)$ be an effective domain and let $F : D \to \text{DOM}$ be a continuous functor. Then $F$ is *effective* if $F(\alpha(n))$ is an effective domain uniformly in $n \in \Omega_\alpha$, and if $\alpha(m) \sqsubseteq \alpha(n)$ then $(F^+[\alpha(m), \alpha(n)], F^-[\alpha(m), \alpha(n)])$ is an effective projection pair from $F(\alpha(m))$ to $F(\alpha(n))$, uniformly in $m, n \in \Omega_\alpha$.

The uniformity is crucial in the definition. It means that there is a recursive function which given $n \in \Omega_\alpha$ computes the computability machinery for $F(\alpha(n))_c$, i.e. a tuple with indices for the code set, the bottom element, the Cons and equality relations and the supremum operation. Similarly for the projection pair. Note that by Proposition 3.4 we could replace the effectivity requirement on the projection pair by requiring that $F^+[\alpha(m), \alpha(n)] \upharpoonright F(\alpha(m))_c$ be computable, uniformly in $m$ and $n$.

If $F : D \to \text{DOM}$ is effective then $F(a)$ is an effective domain for each $a \in D_c$. The uniformity requirement gives us more, namely that $F(x)$ is effective whenever $x$ is effective, uniformly in an index for $x$ (an r.e. index for $\alpha^{-1}(\text{approx}(x))$). This result is analogous to Theorem 3.3(i). To prove it we need the following observation.

**Lemma 3.6.** *Let $F : D \to \text{DOM}$ be a continuous functor. Let $x \in D$, $a, b \in \text{approx}(x)$, $d \in F(a)_c$, $e \in F(b)_c$ and put $c = a \sqcup b$. Then*

(i) $d^{(x)} = e^{(x)} \Leftrightarrow d^{(c)} = e^{(c)}$,

(ii) $\text{Cons}_{F(x)}(d^{(x)}, e^{(x)}) \Leftrightarrow \text{Cons}_{F(c)}(d^{(c)}, e^{(c)})$, *and*

(iii) *if* $\text{Cons}_{F(x)}(d^{(x)}, e^{(x)})$ *then* $d^{(x)} \sqcup_{F(x)} e^{(x)} = (d^{(c)} \sqcup_{F(c)} e^{(c)})^{(x)}$.

**Theorem 3.7.** *Let $F: D \to \mathrm{DOM}$ be an effective continuous functor. If $x \in D$ is effective then $F(x)$ is an effective domain, uniformly in $x$.*

**Proof.** Recall that $F(x)_c = \bigcup \{F(a)_c^{(x)} : a \in \mathrm{approx}(x)\}$ since $F$ is continuous. Let $\alpha$ be the effective numbering of $D$, let $\Omega_n$ be the index set for $F(\alpha(n))$ and let $[\ ]_n : \Omega_n \to F(\alpha(n))_c$ be the coding function for $F(\alpha(n))$, i.e. $[m]_n$ is the $m$th element in $F(\alpha(n))_c$. Let $\Omega_\beta = \{\langle m, n\rangle : m \in \Omega_n \ \&\ \alpha(n) \sqsubseteq x\}$. Then $\Omega_\beta$ is an r.e. set. Define $\beta : \Omega_\beta \to F(x)_c$ by

$$\beta(\langle m, n\rangle) = [m]_n^{(x)}.$$

Then $\beta$ is a surjection. We show that $(F(x), \beta)$ is an effective domain.

Given $\langle m_1, n_1\rangle$ and $\langle m_2, n_2\rangle \in \Omega_\beta$ let $d = [m_1]_{n_1}$ and $e = [m_2]_{n_2}$. Compute $n \in \Omega_\alpha$ such that $\alpha(n) = \alpha(n_1) \sqcup \alpha(n_2)$. By Proposition 3.4 compute indices for $d^{(\alpha(n))}$ and $e^{(\alpha(n))}$ in $\Omega_n$ and decide equality of these elements there. This provides an algorithm for equality in $(F(x)_c, \beta)$, by Lemma 3.6(i). The $\beta$-decidability of the Cons relation and the $\beta$-computability of the supremum function is established similarly. Finally to obtain a designated index in $\Omega_\beta$ for $\perp_{F(x)}$ choose $\langle m, n\rangle$ where $n$ is the designated index for $\perp_D$ and $m$ is the designated index for $\perp_{F(\alpha(n))}$. The uniformity claim is clear. □

We can also extend the effectivity of projection pairs to effective elements.

**Lemma 3.8.** *Let $F: D \to \mathrm{DOM}$ be an effective continuous functor and let $x \in D$ be effective. Then for $a \in \mathrm{approx}(x)$, $F_{ax}^{+} \restriction F(a)_c : F(a)_c \to F(x)_c$ is computable, uniformly in $a$ and $x$.*

**Proof.** Using the notation from the proof of Theorem 3.7 let $\alpha(n) = a$ and define $\sigma : \Omega_n \to \Omega_\beta$ by $\sigma(m) = \langle n, m\rangle$. Then $\sigma$ tracks $F_{ax}^{+} \restriction F(a)_c$. □

Consider the above theorem and suppose that $x \in D_c$. Then the numbering for $F(x)$ obtained from the theorem usually differs from the numbering given by $F$. However, it is not hard to show that these numberings are recursively equivalent.

**Lemma 3.9.** *Let $F: D \to \mathrm{DOM}$ be an effective continuous functor and let $x, y \in D$ be effective such that $x \sqsubseteq y$. Then $F_{xy}^{+} \restriction F(x)_c : F(x)_c \to F(y)_c$ is computable, uniformly in $x$ and $y$.*

**Proof.** Let $b \in F(x)_c$. We shall compute $b^{(y)}$. Search computably for $a \in \mathrm{approx}(x)$ and $b' \in F(a)_c$ such that $b'^{(x)} = b$. The computability of the search follows from the hypotheses and Lemma 3.8, and the termination follows from the continuity of $F$. Then

$$b^{(y)} = b'^{(x)(y)} = b'^{(y)}$$

and the latter is computable, again by Lemma 3.8. The uniformity is immediate. □

The above results show that an effective continuous functor $F: D \to \mathrm{DOM}$ is computable on all of $D_k = \{x \in D : x \text{ effective}\}$, the constructive part of $D$, and not just on $D_c$. This is an analogue to the result that the restriction of an effective function $f: D \to E$ to $D_k$ is computable.

Consider the simple examples given in Section 1. Clearly the constant parametrization of Example 1.10 is effective in case $D$ and $E$ are effective domains. The initial segment functor $F: D \to \mathrm{DOM}$ of Example 1.11 defined by $F(x) = D^x$ is also effective in case $D$ is an effective domain. For given $a \in D_c$, $(D^a)_c = \{b \in D_c : b \sqsubseteq a\}$ and $b \sqsubseteq a$ is decidable. Furthermore the projection embedding is just the inclusion map and its restriction to $(D^a)_c$ is clearly computable.

In order to obtain an effective interpretation of intuitionistic type theory we need to show that the operations on parametrizations given in the previous sections are effective. First we consider the effective version of Proposition 1.13.

**Proposition 3.10.** *Let $F: E \to \mathrm{DOM}$ be an effective continuous functor and let $f: D \to E$ be an effective continuous function. Then $G: D \to \mathrm{DOM}$ is an effective continuous functor, where $G(x) = F(f(x))$ for $x \in D$, and $G(i_x{}^y) = F(i_{f(x)}{}^{f(y)})$ when $x \sqsubseteq y$.*

**Proof.** We just need to be concerned with the effectivity. Let $a \in D_c$. Then $f(a)$ is effective in $E$, uniformly in $a$, and hence $G(a) = F(f(a))$ is effective, uniformly in $a$, by Theorem 3.7. Suppose $a, b \in D_c$ and $a \sqsubseteq b$. Then $f(a) \sqsubseteq f(b)$ and both are effective, uniformly in $a$ and $b$, and hence $G^+_{ab} = F^+_{f(a)f(b)}$ and $G^-_{ab} = F^-_{f(a)f(b)}$ are effective uniformly in $a$ and $b$ by Lemma 3.9 and 3.4. □

The effectiveness of the disjoint sum construction is contained in the following observation, whose proof is immediate.

**Lemma 3.11.** *Let $F: D \to \mathrm{DOM}$ be a continuous functor and suppose $(a, b), (d, e) \in \Sigma(D, F)_c$. Then*

(i) $(a, b) = (d, e) \Leftrightarrow a = d$ *in $D$ and $b = e$ in $F(a)_c$.*

(ii) $\mathrm{Cons}_{\Sigma(D,F)}((a, b), (d, e)) \Leftrightarrow \mathrm{Cons}_D(a, d) \,\&\, \mathrm{Cons}_{F(c)}(b^{(c)}, e^{(c)})$ *where* $c = a \sqcup_D d$.

(iii) $(a, b) \sqcup_{\Sigma(D,F)} (d, e) = (a \sqcup_D d, b^{(a \sqcup_D d)} \sqcup e^{(a \sqcup_D d)})$.

**Theorem 3.12.** *Let $F: D \to \mathrm{DOM}$ be an effective parametrization. Then $\Sigma(D, F)$ is an effective domain.*

**Proof.** Recall that $\Sigma(D, F)_c = \{(a, b) : a \in D_c, b \in F(a)_c\}$. Suppose $\alpha$ is the effective numbering of $D$. Let $\Omega_n$ be the index set for $F(\alpha(n))$ and let

$[\ ]_n : \Omega_n \to F(\alpha(n))_c$ be the coding function for $F(\alpha(n))$. Define $\Omega_\beta = \{\langle m, n\rangle : n \in \Omega_\alpha, m \in \Omega_n\}$ and then define $\beta : \Omega_\beta \to \Sigma(D, F)_c$ by $\beta(\langle m, n\rangle) = (\alpha(n), [m]_n)$. Then $(\Sigma(D, F), \beta)$ is an effective domain by Lemma 3.11. □

Next we consider the cartesian product.

**Definition 3.13.** Let $F : D \to \mathrm{DOM}$ be an effective continuous functor and suppose that $f$ is a p-continuous function w.r.t. $F$. Then $f$ is said to be *effective* if $\mathrm{approx}_{F(a)}(f(a))$ is a semidecidable subset of $F(a)_c$, uniformly in $a \in D_c$.

A p-continuous function $f$ may be identified with its graph, a subset of $\Sigma(D, F)$. The following equivalence is left to the reader.

**Proposition 3.14.** *Let $f$ be p-continuous w.r.t. $F$, where $F$ is effective. Then $f$ is effective if and only if $\mathscr{C}_f = \{(a, b) \in \Sigma(D, F)_c : b \sqsubseteq f(a)\}$ is semidecidable.*

**Proposition 3.15.** *Let $F : D \to \mathrm{DOM}$ be an effective parametrization and suppose $f$ is p-continuous w.r.t. $F$. If $x \in D$ is effective then $f(x) \in F(x)$ is effective.*

**Proof.** We show that

$$\mathrm{approx}_{F(x)}(f(x)) = \{b^{(x)} : a \in \mathrm{approx}(x) \;\&\; (a, b) \in \mathscr{C}_f\}.$$

Previous lemmata assure that the right-hand side is semidecidable. The inclusion from right to left is just the p-monotonicity of $f$. For the converse inclusion suppose $b \in F(x)_c$ and $b \sqsubseteq f(x)$. By p-continuity, $f(x) = \bigsqcup \{f(a)^{(x)} : a \in \mathrm{approx}(x)\}$ so $b \sqsubseteq f(a')^{(x)}$ for some $a' \in \mathrm{approx}(x)$. Furthermore, by the continuity of $F$ there is $a'' \in \mathrm{approx}(x)$ and $b'' \in F(a'')_c$ such that $b''^{(x)} = b$. Let $a = a' \sqcup a''$ and $\bar{b} = b''^{(a)}$. Then it is easily seen that $(a, \bar{b}) \in \mathscr{C}_f$ and $\bar{b}^{(x)} = b$. □

We shall now show that $\Pi(D, F)$ is an effective domain when $F : D \to \mathrm{DOM}$ is an effective parametrization. We use the notation established in Section 2. The proof of the following lemma is essentially contained in the proof of Theorem 2.10.

**Lemma 3.16.** (i) *The set $\{\langle a_i; b_i\rangle : (a_i, b_i) \in \Sigma(D, F)_c,\ i = 1, \ldots, n\}$ is consistent in $\Pi(D, F)$ if and only if for each $I \subseteq \{1, \ldots, n\}$, if $\{a_i : i \in I\}$ is consistent in $D$ then $\{F^+[a_i, \bigsqcup \{a_i : i \in I\}](b_i)\}$ is consistent in $F(\bigsqcup \{a_i : i \in I\})$.*

(ii) *Suppose $A = \{\langle a_i; b_i\rangle : (a_i, b_i) \in \Sigma(D, F)_c,\ i = 1, \ldots, n\}$ is consistent. Then*

$$(\bigsqcup A)(x) = \bigsqcup \{b_i^{(x)} : i \in I_x\}$$

*where $I_x = \{i : a_i \sqsubseteq x\}$.*

**Theorem 3.17.** *Let $F : D \to \mathrm{DOM}$ be an effective parametrization. Then $\Pi(D, F)$ is an effective domain.*

**Proof.** Recall that $\Pi(D, F)_c$ consists of all suprema of consistent sets of the form $\{\langle a_i; b_i\rangle : (a_i, b_i) \in \Sigma(D, F)_c, i = 1, \ldots, n\}$. Suppose $\alpha$ is the effective numbering of $D$. Let $\Omega_n$ be the index set for $F(\alpha(n))$ and let $[\ ]_n : \Omega_n \to F(\alpha(n))_c$ be the coding function for $F(\alpha(n))$. Let $\beta$ be the effective numbering of $\Sigma(D, F)$ obtained in Theorem 3.12, and let $\delta : \Omega_\beta \to \Pi(D, F)_c$ be defined by $\delta(\langle m, n\rangle) = \langle \alpha(n); [m]_n\rangle$. Then, identifying finite sets of numbers with canonical indices for these sets in some standard way, define

$$\Omega_\gamma = \{K \subseteq \Omega_\beta : \delta(K) \text{ is consistent}\}$$

and define $\gamma : \Omega_\gamma \to \Pi(D, F)_c$ by

$$\gamma(K) = \bigsqcup \delta(K).$$

Note that $\Omega_\gamma$ is r.e. by Lemma 3.16(i).

Let $K \in \Omega_\gamma$. Then for $e \in D_e$,

$$\gamma(K)(e) = \bigsqcup \{([j]_i)^{(e)} : \alpha(i) \sqsubseteq e, \langle i, j\rangle \in K\}$$

which is computable, since $K$ is finite. Furthermore, for any $g \in \Pi(D, F)$,

$$\gamma(K) \sqsubseteq g \quad \Leftrightarrow \quad \gamma(K)(\alpha(i)) \sqsubseteq g(\alpha(i)) \text{ for each } i \in \pi_0(K).$$

It follows that, for $K, L \in \Omega_\gamma$, $\gamma(K) = \gamma(L)$ is decidable. Finally Lemma 3.16 shows how to decide consistency and how to compute suprema. $\square$

The following proposition is an easy exercise.

**Proposition 3.18.** *Let $F : D \to \mathrm{DOM}$ be an effective parametrization and let $f$ be p-continuous w.r.t. $F$. Then $f$ is effective if and only if $f$ is effective as an element of $\Pi(D, F)$.*

We now consider the effectivity of $\Sigma$ and $\Pi$ as functors.

**Definition 3.19.** Let $F : D \to \mathrm{DOM}$ and $G : \Sigma(D, F) \to \mathrm{DOM}$ be continuous functors. Then $G$ is an *effective parametrization over* $F$ if $F$ is effective and $G$ is effective over the *standard numbering* of $\Sigma(D, F)$ obtained by Theorem 3.12.

Let $G$ be an effective parametrization over $F$. Fix $a \in D_c$ and define $f_a : F(a) \to \Sigma(D, F)$ by $f_a(y) = (a, y)$. Then $f_a$ is continuous and effective, uniformly in $a$. Consider the parametrization $(y)G(a, y) : F(a) \to \mathrm{DOM}$. Then $(y)G(a, y) = (y)G(f_a(y))$ and the latter is effective by Proposition 3.10.

**Theorem 3.20.** *Let $G$ be an effective parametrization over $F$. Then $\Sigma(F, G) : D \to \mathrm{DOM}$ is an effective parametrization.*

**Proof.** Denote $\Sigma(F, G)$ by $H$. Recall that $H(a) = \Sigma(F(a), (y)G(a, y))$ for $a \in D_c$. By the effectivity of $F$ and $G$, the remark above and by the uniformity of

Theorem 3.12, $H(a)$ is an effective domain uniformly in $a$. Suppose $a, b \in D_c$ and $a \sqsubseteq b$, and let $(d, e) \in H(a)_c$. Then

$$H^+_{ab}(d, e) = (d^{(b)}, e^{(b, d^{(b)})})$$

Each component is computable, and hence $H^+_{ab}$ restricted to $H(a)_c$ is computable. The uniformity in $a$ and $b$ is immediate. □

**Theorem 3.21.** *Let $G$ be an effective parametrization over $F$. Then $\Pi(F, G): D \to \mathrm{DOM}$ is an effective parametrization.*

**Proof.** Denote $\Pi(F, G)$ by $H$. By the uniformity of Theorem 3.17, $H(a)$ is an effective domain uniformly in $a \in D_c$. Let $a, b \in D_c$ and suppose $a \sqsubseteq b$. Consider a basic compact function $\langle d; e\rangle \in H(a)_c$, i.e. $d \in F(a)_c$ and $e \in G(a, d)_c$. Let $d' = d^{(b)}$ and let $e' = e^{(b, d')}$. Then $H^+_{ab}(\langle d; e\rangle) = \langle d'; e'\rangle$. Thus to compute $H^+_{ab}(\langle d; e\rangle)$ we just compute $d'$ and $e'$ which is possible by assumption. Of course, the computation is uniform in $a$ and $b$. □

We leave to the reader to verify that the remaining results in Sections 1 and 2, such as Theorems 2.21 and 2.22, have their effective counterparts.

## 4. Partial intuitionistic type theory

In this section we present, as a formal system, the version of Martin-Löf's intuitionistic type theory that we shall interpret.

*Monomorphic* type theory is the version of type theory where type information is attached to terms. *Polymorphic* type theory is obtained from monomorphic type theory by stripping type information from all proper part of terms. It is a polymorphic version of type theory that is presented in Martin-Löf [9]. However, it is the monomorphic theory which is the intended type theory. Results in Salvesen [21] indicate that the distinction between polymorphic and monomorphic type theories is non-trivial and that the theories are fundamentally different. The effect of choosing a monomorphic version of type theory is to restrict possible derivations since, essentially, these are coded into the terms. This is necessary for us in order to prove that our interpretation is well-defined, that is independent of formal derivations.

As a basic reference for type theory we use Martin-Löf [9]; also Troelstra [20] contains useful information. However, the system presented here differs from the one in Martin-Löf [9], and also Troelstra [20], in a few respects. The important difference is that we use a weaker identity type (also due to Martin-Löf), which is essential for our interpretation. It should be noted that Martin-Löf regards the weaker identity type as the correct one to use. Indeed, partial type theory trivializes with the extensional identity type in the sense that all elements of a type become equal. Another difference from Martin-Löf [9] is that we use a sequent notation in order to keep the assumptions (contexts) explicit. Furthermore, derivations in our system are sequences of statements (judgements) rather than trees as in the natural deduction system of Martin-Löf [9]. We also include the $\Omega$ type to obtain partial type theory. It is the $\Omega$ type which provides a main motivation for giving a domain interpretation of type theory.

The statements which occur in (our formal system of) intuitionistic type theory are called *judgements*. They come in one of five forms:

$\Gamma$ **context**,

$\Gamma \Rightarrow A$ **set**,

$\Gamma \Rightarrow a \in A$,

$\Gamma \Rightarrow A = B$,

$\Gamma \Rightarrow a = b \in A$.

The judgements are built up from an infinite set of variables and a set of constants in the way prescribed by the rules of the formal system. Thus a syntactic expression $\Gamma$ **context** or $\Gamma \Rightarrow \Theta$ is a judgement if and only if it has been derived in the formal system.

In a judgement $\Gamma$ **context** or $\Gamma \Rightarrow \Theta$, the part $\Gamma$ is called a *context*. As is apparent from the rules below, a context will have the form

$$x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n$$

where $x_1, \ldots, x_n$ are distinct variables such that the variable $x_i$ does not appear free in $A_j$ for $j \leqslant i$. The empty sequence is denoted by $\emptyset$. It follows from the context rule below that $\emptyset$ **context** is a judgement, i.e. that $\emptyset$ is a context.

Traditionally, in expositions of intuitionsitic type theory, the rules are presented without making all *presuppositions* of the rules explicit. The presuppositions of $\Gamma$ **context**, where $\Gamma \equiv x_1 \in A_1,\ x_2 \in A_2, \ldots, x_n \in A_n$ ($\equiv$ denotes syntactic equality) are $\emptyset \Rightarrow A_1$ **set**, $x_1 \in A_1 \Rightarrow A_2$ **set**, $\ldots$, and $x_1 \in A_1, x_2 \in A_2, \ldots, x_{n-1} \in A_{n-1} \Rightarrow A_n$ **set**. The presupposition of $\Gamma \Rightarrow A$ **set** is $\Gamma$ **context**. Similarly, the presupposition of $\Gamma \Rightarrow a \in A$ is $\Gamma \Rightarrow A$ **set**, the presuppositions of $\Gamma \Rightarrow A = B$ are $\Gamma \Rightarrow A$ **set** and $\Gamma \Rightarrow B$ **set**, and the presuppositions of $\Gamma \Rightarrow a = b \in A$ are $\Gamma \Rightarrow a \in A$ and $\Gamma \Rightarrow b \in A$.

In order to reflect the required presuppositions of the rules, we *stipulate*, in addition to the rules given below, the following general requirements for the formal system: In order to derive $\Gamma \Rightarrow A$ **set** we must first have derived $\Gamma$ **context**, i.e. $\Gamma$ **context** must appear in the derivation before $\Gamma \Rightarrow A$ **set**. Similarly, in order to derive $\Gamma \Rightarrow a \in A$ we must first have derived $\Gamma \Rightarrow A$ **set**, in order to derive $\Gamma \Rightarrow A = B$ we must first have derived $\Gamma \Rightarrow A$ **set** and $\Gamma \Rightarrow B$ **set**, and, finally, in order to derive $\Gamma \Rightarrow a = b \in A$ we must first have derived $\Gamma \Rightarrow a \in A$ and $\Gamma \Rightarrow b \in A$ (and hence $\Gamma \Rightarrow A$ **set** and $\Gamma$ **context**).

We now list the rules of the system. Recall that derivations are sequences of judgements and that in order to use a rule the stipulation given above must first be met. In the usual manner we identify expressions which only differ among bound variables. By $A(a_1, \ldots, a_n/x_1, \ldots, x_n)$ we mean the simultaneous substitution of the expressions $a_1, \ldots, a_n$ into the expression $A$, where each free occurrence of the variable $x_i$ is replaced by $a_i$. Of course, whenever we make a substitution it is assumed that the $a_i$ are substitutable, i.e. no variable occurring freely in $a_i$ becomes bound in $A(a_1, \ldots, a_n/x_1, \ldots, x_n)$. The variables $x_1, \ldots, x_n$ become bound in an expression of the form $(x_1, \ldots, x_n)$exp where $x_i$ is a variable and exp is an expression.

*Context rules*

$$\emptyset\ \textbf{context} \qquad\qquad \frac{\Gamma \Rightarrow A\ \textbf{set}}{\Gamma, x \in A\ \textbf{context}}$$

provided $x$ does not appear free in $\Gamma$.

*Assumption rule*

$$x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n \;\Rightarrow\; x_i \in A_i$$

for $1 \leq i \leq n$.

*Equality rules*

Reflexivity:

$$\frac{\Gamma \Rightarrow a \in A}{\Gamma \Rightarrow a = a \in A} \qquad \frac{\Gamma \Rightarrow A\ \textbf{set}}{\Gamma \Rightarrow A = A}$$

Symmetry:

$$\frac{\Gamma \Rightarrow a = b \in A}{\Gamma \Rightarrow b = a \in A} \qquad \frac{\Gamma \Rightarrow A = B}{\Gamma \Rightarrow B = A}$$

Transitivity:

$$\frac{\begin{array}{c}\Gamma \Rightarrow a = b \in A \\ \Gamma \Rightarrow b = c \in A\end{array}}{\Gamma \Rightarrow a = c \in A} \qquad \frac{\begin{array}{c}\Gamma \Rightarrow A = B \\ \Gamma \Rightarrow B = C\end{array}}{\Gamma \Rightarrow A = C}$$

Equality of sets:

$$\frac{\begin{array}{c}\Gamma \Rightarrow a \in A \\ \Gamma \Rightarrow A = B\end{array}}{\Gamma \Rightarrow a \in B} \qquad \frac{\begin{array}{c}\Gamma \Rightarrow a = b \in A \\ \Gamma \Rightarrow A = B\end{array}}{\Gamma \Rightarrow a = b \in B}$$

*Cartesian product of a family of sets*

Π-formation:

$$\frac{\Gamma, x \in A \Rightarrow B \textbf{ set}}{\Gamma \Rightarrow \Pi(A, (x)B) \textbf{ set}}$$

$$\frac{\Gamma \Rightarrow A = A' \qquad \Gamma, x \in A \Rightarrow B = B'}{\Gamma \Rightarrow \Pi(A, (x)B) = \Pi(A', (x)B')}$$

Π-introduction:

$$\frac{\Gamma, x \in A \Rightarrow b \in B}{\Gamma \Rightarrow \lambda(A, (x)B, (x)b) \in \Pi(A, (x)B)}$$

$$\frac{\Gamma, x \in A \Rightarrow b = b' \in B}{\Gamma \Rightarrow \lambda(A, (x)B, (x)b) = \lambda(A, (x)B, (x)b')}$$

Π-elimination:

$$\frac{\Gamma \Rightarrow c \in \Pi(A, (x)B) \qquad \Gamma \Rightarrow a \in A}{\Gamma \Rightarrow \mathrm{Ap}(A, (x)B, c, a) \in B(a/x)}$$

$$\frac{\Gamma \Rightarrow c = c' \in \Pi(A, (x)B) \qquad \Gamma \Rightarrow a = a' \in A}{\Gamma \Rightarrow \mathrm{Ap}(A, (x)B, c, a) = \mathrm{Ap}(A, (x)B, c', a') \in B(a/x)}$$

Π-equality:

$$\frac{\Gamma \Rightarrow a \in A \qquad \Gamma, x \in A \Rightarrow b \in B}{\Gamma \Rightarrow \mathrm{Ap}(A, (x)B, \lambda(A, (x)B, (x)b), a) = b(a/x) \in B(a/x)}$$

There is a rule of equality for each formation, introduction and elimination rule analogous to the ones given above for the cartesian product. In the sequel, in order to save space, these rules will not be written out.

*Disjoint union of a family of sets*

Σ-formation:

$$\frac{\Gamma, x \in A \Rightarrow B \textbf{ set}}{\Gamma \Rightarrow \Sigma(A, (x)B) \textbf{ set}}$$

Σ-introduction:

$$\frac{\Gamma \Rightarrow a \in A \qquad \Gamma \Rightarrow b \in B(a/x)}{\Gamma \Rightarrow p(A, (x)B, a, b) \in \Sigma(A, (x)B)}$$

Σ-elimination:

$$\frac{\begin{array}{c}\Gamma,\ z \in \Sigma(A,\ (x)B) \Rightarrow C \textbf{ set} \\ \Gamma \Rightarrow c \in \Sigma(A,\ (x)B) \\ \Gamma,\ x \in A,\ y \in B \Rightarrow d \in C(p(A,\ (x)B,\ x,\ y)/z)\end{array}}{\Gamma \Rightarrow E(A,\ (x)B,\ (z)C,\ c,\ (x,\ y)d) \in C(c/z)}$$

Σ-equality:

$$\frac{\begin{array}{c}\Gamma,\ z \in \Sigma(A,\ (x)B) \Rightarrow C \textbf{ set} \\ \Gamma \Rightarrow a \in A \\ \Gamma \Rightarrow b \in B(a/x) \\ \Gamma,\ x \in A,\ y \in B \Rightarrow d \in C(p(A,\ (x)B,\ x,\ y)/z)\end{array}}{\begin{array}{c}\Gamma \Rightarrow E(A,\ (x)B,\ (z)C,\ p(A,\ (x)B,\ a,\ b),\ (x,\ y)d) \\ = d(a,\ b/x,\ y) \in C(p(A,\ (x)B,\ a,\ b)/z)\end{array}}$$

*Disjoint union of two sets*

+-formation:

$$\frac{\begin{array}{c}\Gamma \Rightarrow A \textbf{ set} \\ \Gamma \Rightarrow B \textbf{ set}\end{array}}{\Gamma \Rightarrow A + B \textbf{ set}}$$

+-introduction:

$$\frac{\Gamma \Rightarrow a \in A}{\Gamma \Rightarrow i(A,\ B,\ a) \in A + B} \qquad \frac{\Gamma \Rightarrow b \in B}{\Gamma \Rightarrow j(A,\ B,\ b) \in A + B}$$

+-elimination:

$$\frac{\begin{array}{c}\Gamma,\ z \in A + B \Rightarrow C \textbf{ set} \\ \Gamma \Rightarrow c \in A + B \\ \Gamma,\ x \in A \Rightarrow d \in C(i(A,\ B,\ x)/z) \\ \Gamma,\ y \in B \Rightarrow e \in C(j(A,\ B,\ y)/z)\end{array}}{\Gamma \Rightarrow D(A,\ B,\ (z)C,\ c,\ (x)d,\ (y)e) \in C(c/z)}$$

+-equality:

$$\frac{\begin{array}{c}\Gamma,\ z \in A + B \Rightarrow C \textbf{ set} \\ \Gamma \Rightarrow a \in A \\ \Gamma,\ x \in A \Rightarrow d \in C(i(A,\ B,\ x)/z) \\ \Gamma,\ y \in B \Rightarrow e \in C(j(A,\ B,\ y)/z)\end{array}}{\Gamma \Rightarrow D(A,\ B,\ (z)C,\ i(A,\ B,\ a),\ (x)d,\ (y)e) = d(a/x) \in C(i(A,\ B,\ a)/z)}$$

and

$$\frac{\begin{array}{c}\Gamma,\, z \in A + B \Rightarrow C \textbf{ set} \\ \Gamma \Rightarrow b \in B \\ \Gamma,\, x \in A \Rightarrow d \in C(i(A, B, x)/z) \\ \Gamma,\, y \in B \Rightarrow e \in C(j(A, B, y)/z)\end{array}}{\Gamma \Rightarrow D(A, B, (z)C, j(A, B, b), (x)d, (y)e) = e(b/y) \in C(j(A, B, b)/z)}$$

*Identity relation*

$I$-formation:

$$\frac{\begin{array}{c}\Gamma \Rightarrow a \in A \\ \Gamma \Rightarrow b \in A\end{array}}{\Gamma \Rightarrow I(A, a, b) \textbf{ set}}$$

$I$-introduction:

$$\frac{}{\Gamma \Rightarrow r(A, a) \in I(A, a, a)}$$

$I$-elimination:

$$\frac{\begin{array}{c}\Gamma,\, x \in A,\, y \in A,\, z \in I(A, x, y) \Rightarrow C \textbf{ set} \\ \Gamma \Rightarrow c \in I(A, a, b) \\ \Gamma,\, x \in A \Rightarrow d \in C(x, r(A, x)/y, z)\end{array}}{\Gamma \Rightarrow J(A, (x, y, z)C, c, (x)d) \in C(a, b, c/x, y, z)}$$

$I$-equality:

$$\frac{\begin{array}{c}\Gamma,\, x \in A,\, y \in A,\, z \in I(A, x, y) \Rightarrow C \textbf{ set} \\ \Gamma \Rightarrow a \in A \\ \Gamma,\, x \in A \Rightarrow d \in C(x, r(A, x)/y, z)\end{array}}{\Gamma \Rightarrow J(A, (x, y, z)C, r(A, a), (x)d) = d(a/x) \in C(a, a, r(A, x)/x, y, z)}$$

*Finite sets*

$N_n$-formation (for $n = 0, 1, 2, \ldots$):

$$\Gamma \Rightarrow N_n \textbf{ set}$$

$N_n$-introduction

$$\Gamma \Rightarrow m_n \in N_n \quad \text{for } m = 0, \ldots, n-1.$$

$N_n$-elimination:

$$\frac{\begin{array}{c}\Gamma,\, z \in N_n \Rightarrow C \textbf{ set} \\ \Gamma \Rightarrow c \in N_n \\ \Gamma \Rightarrow c_i \in C(i_n/z) \text{ for } i = 0, \ldots, n-1\end{array}}{\Gamma \Rightarrow R_n((z)C, c, c_0, \ldots, c_{n-1}) \in C(c/z)}$$

$N_n$-equality

$$\frac{\Gamma, z \in N_n \Rightarrow C \ \mathbf{set} \qquad \Gamma \Rightarrow c_i \in C(i_n/z) \text{ for } i = 0, \ldots, n-1}{\Gamma \Rightarrow R_n((z)C, m_n, c_0, \ldots, c_{n-1}) = c_m \in C(c/z)}$$

for $m = 0, \ldots, n-1$.

*The set of natural numbers*

$N$-formation:

$$\Gamma \Rightarrow N \ \mathbf{set}$$

$N$-introduction:

$$\Gamma \Rightarrow 0 \in N \qquad \frac{\Gamma \Rightarrow a \in N}{\Gamma \Rightarrow s(a) \in N}$$

$N$-elimination:

$$\frac{\begin{array}{c}\Gamma, z \in N \Rightarrow C \ \mathbf{set} \\ \Gamma \Rightarrow c \in N \\ \Gamma \Rightarrow d \in C(0/z) \\ \Gamma, x \in N, y \in C(x/z) \Rightarrow e \in C(s(x)/z)\end{array}}{\Gamma \Rightarrow R((z)C, c, d, (x, y)e) \in C(c/z)}$$

$N$-equality:

$$\frac{\begin{array}{c}\Gamma, z \in N \Rightarrow C \ \mathbf{set} \\ \Gamma \Rightarrow d \in C(0/z) \\ \Gamma, x \in N, y \in C(x/z) \Rightarrow e \in C(s(x)/z)\end{array}}{\Gamma \Rightarrow R((z)C, 0, d, (x, y)e) = d \in C(0/z)}$$

and

$$\frac{\begin{array}{c}\Gamma, z \in N \Rightarrow C \ \mathbf{set} \\ \Gamma \Rightarrow a \in N \\ \Gamma \Rightarrow d \in C(0/z) \\ \Gamma, x \in N, y \in C(x/z) \Rightarrow e \in C(s(x)/z)\end{array}}{\Gamma \Rightarrow R((z)C, s(a), d, (x, y)e) = e(a, R((z)C, a, d, (x, y)e)/x, y) \in C(s(a)/z)}$$

*The iteration set*

$\Omega$-formation:

$$\Gamma \Rightarrow \Omega \ \mathbf{set}$$

$\Omega$-introduction:

$$\frac{\Gamma \Rightarrow a \in \Omega}{\Gamma \Rightarrow a' \in \Omega}$$

$\Omega$-elimination:

$$\frac{\begin{array}{c}\Gamma,\ z \in \Omega \Rightarrow C\ \mathbf{set} \\ \Gamma \Rightarrow c \in \Omega \\ \Gamma,\ x \in \Omega,\ y \in C(x/z) \Rightarrow d \in C(x'/z)\end{array}}{\Gamma \Rightarrow R_\omega((z)C,\ c,\ (x,\ y)d) \in C(c/z)}$$

$\Omega$-equality:

$$\frac{\begin{array}{c}\Gamma,\ z \in \Omega \Rightarrow C\ \mathbf{set} \\ \Gamma \Rightarrow a \in \Omega \\ \Gamma,\ x \in \Omega,\ y \in C(x/z) \Rightarrow d \in C(x'/z)\end{array}}{\Gamma \Rightarrow R_\omega((z)C,\ a',\ (x,\ y)d) = d(a,\ R_\omega((z)C,\ a,\ (x,\ y)d)/x,\ y) \in C(a'/z)}$$

Additional axioms for $\omega$

$$\Gamma \Rightarrow \omega \in \Omega \qquad\qquad \Gamma \Rightarrow \omega = \omega' \in \Omega.$$

This completes the list of rules for the monomorphic system that we consider.

The polymorphic theory is obtained from the monomorphic theory by a *stripping* operation * defined inductively on expressions for types and for terms.

$$\Pi(A,\ (x)B)^* = (\Pi x \in A^*)B^* \qquad N_n^* = N_n$$
$$\lambda(A,\ (x)B,\ (x)b)^* = (\lambda x)b^* \qquad m_n^* = m_n$$
$$\mathrm{Ap}(A,\ (x)B,\ c,\ a)^* = \mathrm{Ap}(c^*,\ a^*) \qquad R_n((z)C,\ c,\ c_0,\ \ldots,\ c_{n-1})^* = R_n(c^*,\ c_0^*,\ \ldots,\ c_{n-1}^*)$$

$$\Sigma(A,\ (x)B)^* = (\Sigma x \in A^*)B^* \qquad N^* = N$$
$$p(A,\ (x)B,\ a,\ b)^* = (a^*,\ b^*) \qquad 0^* = 0 \quad s(a)^* = s(a^*)$$
$$E(A,\ (x)B,\ (z)C,\ c,\ (x,\ y)d)^* = E(c^*,\ (x,\ y)d^*) \qquad R((z)C,\ c,\ d,\ (x,\ y)e)^* = R(c^*,\ d^*,\ (x,\ y)e^*)$$

$$(A + B)^* = A^* + B^* \qquad \Omega^* = \Omega$$
$$i(A,\ B,\ a)^* = i(a^*) \quad j(A,\ B,\ b)^* = j(b^*) \qquad \omega^* = \omega \quad (a')^* = (a^*)'$$
$$D(A,\ B,\ (z)C,\ c,\ (x)d,\ (y)e)^* = D(c^*,\ (x)d^*,\ (y)e^*) \qquad R_\omega((z)C,\ c,\ (x,\ y)e)^* = R_\omega(c^*,\ (x,\ y)e^*)$$

$$I(A,\ a,\ b)^* = I(A^*,\ a^*,\ b^*)$$
$$r(A,\ a)^* = r(a^*)$$
$$J(A,\ (x,\ y,\ z)C,\ c,\ (x)d)^* = J(c^*,\ (x)d^*)$$

The polymorphic theory obtained by * corresponds to the version given in Martin-Löf [9].

In the remaining part of the paper we shall use the polymorphic version of type theory. Note however that this is only a notational convenience; the type information on terms is assumed to be available.

The thinning and substitution rules are derived rules of the system. Their proofs reveal how thinning and substitution are to be interpreted.

**Theorem 4.1.** *The following rules are derived rules of the system, where $w$ is a variable not free in $\Gamma$, $\Delta$:*

(i)
$$\frac{\Gamma \Rightarrow D\ \mathbf{set} \qquad \Gamma, \Delta\ \mathbf{context}}{\Gamma, w \in D, \Delta\ \mathbf{context}}$$

(ii)
$$\frac{\Gamma \Rightarrow D\ \mathbf{set} \qquad \Gamma, \Delta \Rightarrow \Theta}{\Gamma, w \in D, \Delta \Rightarrow \Theta}$$

**Proof.** The proof of (i) and (ii) is by induction on the derivation of $\Gamma, \Delta$ **context** or $\Gamma, \Delta \Rightarrow \Theta$. Consider (i). If $\Delta \equiv \emptyset$ then (i) holds by the context rule. Suppose $\Delta \equiv \Delta', z \in C$. Then, since $\Gamma$, $\Delta$ **context**, there is a shorter derivation of $\Gamma, \Delta' \Rightarrow C$ **set**. By the induction hypothesis, $\Gamma, w \in D, \Delta' \Rightarrow C$ **set** and hence $\Gamma, w \in D, \Delta$ **context** by the context rule.

For (ii) we exemplify by considering the case of $\Pi$-formation. Suppose $\Gamma, \Delta \Rightarrow (\Pi x \in A)B$ **set**. Then, inductively, $\Gamma, w \in D, \Delta \Rightarrow A$ **set** and $\Gamma, w \in D, \Delta, x \in A \Rightarrow B$ **set** and hence $\Gamma, w \in D, \Delta \Rightarrow (\Pi x \in A)B$ **set** by $\Pi$-formation. The remaining cases are similar. $\square$

**Theorem 4.2.** *Assume $\Gamma, \Delta, x_1 \in A_1, \ldots, x_n \in A_n, \Phi$* **context** *and $\Gamma, \Delta \Rightarrow a_1 \in A_1$, $\Gamma, \Delta \Rightarrow a_2 \in A_2(a_1/x_1), \ldots, \Gamma, \Delta \Rightarrow a_n \in A_n(a_1, \ldots, a_{n-1}/x_1, \ldots, x_{n-1})$. Then the following substitution rules are derived rules*:

(i)
$$\frac{\Gamma, x_1 \in A_1, \ldots, x_n \in A_n, \Phi\ \mathbf{context}}{\Gamma, \Delta, \Phi(a_1, \ldots, a_n/x_1, \ldots, x_n)\ \mathbf{context}}$$

(ii)
$$\frac{\Gamma, x_1 \in A_1, \ldots, x_n \in A_n, \Phi \Rightarrow \Theta}{\Gamma, \Delta, \Phi(a_1, \ldots, a_n/x_1, \ldots, x_n) \Rightarrow \Theta(a_1, \ldots, a_n/x_1, \ldots, x_n)}$$

**Remark.** The assumption $\Gamma, \Delta, x_1 \in A_1, \ldots, x_n \in A_n, \Phi$ **context** only makes sure that the variables in $\Phi$ and $\Delta$ are not in conflict with each other.

**Proof.** We use the notation $\boldsymbol{a} \equiv a_1, \ldots, a_n; \boldsymbol{x} \equiv x_1, \ldots, x_n$ and $\boldsymbol{x} \in \boldsymbol{A} \equiv x_1 \in A_1, \ldots, x_n \in A_n$. The proof of (i) and (ii) is by induction on the derivation of $\Gamma, \boldsymbol{x} \in \boldsymbol{A}, \Phi$ **context** or $\Gamma, \boldsymbol{x} \in \boldsymbol{A}, \Phi \Rightarrow \Theta$.

(i): In case $\Phi \equiv \emptyset$ there is nothing to prove, since $\Gamma, \Delta$ **context**. So suppose $\Phi \equiv \Phi', z \in D$. Then we must have $\Gamma, \boldsymbol{x} \in \boldsymbol{A}, \Phi' \Rightarrow D$ **set** with a derivation of shorter length, so inductively, $\Gamma$, $\Delta$, $\Phi'(\boldsymbol{a}/\boldsymbol{x}) \Rightarrow D(\boldsymbol{a}/\boldsymbol{x})$ **set**. *But then* $\Gamma$, $\Delta$, $\Phi(\boldsymbol{a}/\boldsymbol{x})$ **context** by the context rule.

(ii) We only consider the cases of the assumption rule and the Π-formation rule. The remaining rules are similar though sometimes syntactically more complicated. For the assumption rule, assume

$$\Gamma, \boldsymbol{x} \in \boldsymbol{A},\ \Phi \Rightarrow z \in D.$$

By our general stipulation we have $\Gamma, \boldsymbol{x} \in \boldsymbol{A},\ \Phi$ **context** with a derivation of shorter length, so

$$\Gamma,\ \Delta,\ \Phi(\boldsymbol{a}/\boldsymbol{x})\ \textbf{context}$$

by (i). If $z \in D$ comes from $\Gamma$ or $\Phi$ then the result follows by the assumption rule. If $z \in D \equiv x_i \in A_i$ then $(z \in D)(\boldsymbol{a}/\boldsymbol{x}) \equiv a_i \in A_i(a_1, \ldots, a_{i-1}/x_1, \ldots, x_{i-1})$ so

$$\Gamma,\ \Delta,\ \Phi(\boldsymbol{a}/\boldsymbol{x}) \Rightarrow a_i \in A_i(a_1, \ldots, a_{i-1}/x_1, \ldots, x_{i-1})$$

by the thinning rule of Theorem 4.1.

Now consider the Π-formation rule, say we have obtained $\Gamma, \boldsymbol{x} \in \boldsymbol{A},\ \Phi \Rightarrow (\Pi\, y \in B)C$ **set**. Then, with shorter derivations, we have

$$\Gamma, \boldsymbol{x} \in \boldsymbol{A},\ \Phi \Rightarrow B\ \textbf{set} \quad \text{and} \quad \Gamma, \boldsymbol{x} \in \boldsymbol{A},\ \Phi,\ y \in B \Rightarrow C\ \textbf{set}.$$

Inductively, we then have

$$\Gamma,\ \Delta,\ \Phi(\boldsymbol{a}/\boldsymbol{x}) \Rightarrow B(\boldsymbol{a}/\boldsymbol{x})\ \textbf{set} \quad \text{and}$$
$$\Gamma,\ \Delta,\ \Phi(\boldsymbol{a}/\boldsymbol{x}),\ y \in B(\boldsymbol{a}/\boldsymbol{x}) \Rightarrow C(\boldsymbol{a}/\boldsymbol{x})\ \textbf{set}.$$

Thus

$$\Gamma,\ \Delta,\ \Phi(\boldsymbol{a}/\boldsymbol{x}) \Rightarrow (\Pi\, y \in B(\boldsymbol{a}/\boldsymbol{x}))C(\boldsymbol{a}/\boldsymbol{x})\ \textbf{set}$$

by the Π-formation rule. This proves our case since

$$(\Pi\, y \in B(\boldsymbol{a}/\boldsymbol{x}))C(\boldsymbol{a}/\boldsymbol{x}) \equiv ((\Pi\, y \in B)C)(\boldsymbol{a}/\boldsymbol{x}). \qquad \square$$

## 5. The first interpretation

By an interpretation of intuitionistic type theory we mean an interpretation of each judgement of the formal system given in Section 4. An interpretation of a judgement consists of an interpretation of its context and, in case the judgement is of the form $\Gamma \Rightarrow \Theta$, an interpretation of $\Theta$. A context $\Gamma$ is interpreted as an effective domain $[\![\Gamma]\!]$. A type $A$ in a judgement of the form $\Gamma \Rightarrow A$ **set** is interpreted as an effective parametrization $[\![A]\!]_\Gamma : [\![\Gamma]\!] \to \mathrm{DOM}$. A term $a$ in a judgement of the form $\Gamma \Rightarrow a \in A$ is interpreted as an effective p-continuous function $[\![a]\!]_\Gamma$ over $[\![A]\!]_\Gamma$. Thus the symbol $\in$ in a judgement $\Gamma \Rightarrow a \in A$ is interpreted as belonging to the domain $\Pi([\![\Gamma]\!], [\![A]\!]_\Gamma)$. The symbol $=$ in a judgement of the form $\Gamma \Rightarrow A = B$ is interpreted as (set-theoretic) equality between $[\![A]\!]_\Gamma$ and $[\![B]\!]_\Gamma$, that is $[\![A]\!]_\Gamma(w) = [\![B]\!]_\Gamma(w)$ for each $w \in [\![\Gamma]\!]$, and when $w \sqsubseteq v$ in $[\![\Gamma]\!]$ then $[\![A]\!]_\Gamma^+[w, v] = [\![B]\!]_\Gamma^+[w, v]$. Finally, the symbols $=$ and $\in$ in a judgement of the form $\Gamma \Rightarrow a = b \in A$ are interpreted as $[\![a]\!]_\Gamma$ and $[\![b]\!]_\Gamma$ being equal elements of $\Pi([\![\Gamma]\!], [\![A]\!]_\Gamma)$.

The interpretation given in this section is not adequate for the operational semantics of type theory. Instead, for example, the $\eta$-rule holds for the interpretation, that is $[\![(\lambda x)\mathrm{Ap}(c, x)]\!]_\Gamma = [\![c]\!]_\Gamma$ whenever $\Gamma \Rightarrow c \in (\Pi x \in A)B$. In Section 6 we modify the interpretation slightly to obtain an interpretation which is adequate for the operational semantics.

The interpretation of a judgement is given inductively on a derivation of that judgement. Recall the general stipulation made on derivations in Section 4. Thus, if $\Gamma \Rightarrow A$ **set** is a judgement then $\Gamma$ **context** must appear in its derivation and hence the domain $[\![\Gamma]\!]$ is defined inductively. So it makes sense to say that $[\![A]\!]_\Gamma$ shall be a parametrization from $[\![\Gamma]\!]$ into DOM. Similar remarks apply to the remaining forms of judgements.

It is the monomorphic type theory that we shall interpret. Nonethless we use the polymorphic notation when giving the interpretation. Thus the type information on terms is assumed to be there, but it is written with invisible ink made visible only when we need the information.

The interpretation of a type $A$ and a term $a$ will depend only on the context of the judgement in which they appear. This will be proved by induction on derivations. Furthermore we need to show how the interpretation behaves under thinning and substitution (Theorems 4.1 and 4.2). In fact, all of this needs to be shown by induction on derivations simultaneously with giving the interpretation. However, in order to make the presentation more readable, we first give the interpretation and show that all the rules of type theory are satisfied under the assumptions that the interpretations $[\![A]\!]_\Gamma$ and $[\![a]\!]_\Gamma$ depend only on $\Gamma$ and that Theorems 5.1 and 5.2 below hold. Then, at the end of the section, we prove these assumptions.

First we consider the interpretation of a context, after which we state Theorems 5.1 and 5.2 about the interpretations of thinning and substitution. Then we go through the remaining rules of the system, one at a time. To save some writing we assume all inductive assumptions to be effective without explicitly stating this in each case.

*The context rules*

The empty context $\emptyset$ is interpreted by the one point domain, $[\![\emptyset]\!] = \{\bot\}$. Suppose we have derived $\Gamma$, $x \in A$ **context**. Then, by our general stipulation, we have already derived $\Gamma \Rightarrow A$ **set** and $\Gamma$ **context**, and hence, inductively, $[\![\Gamma]\!]$ is a domain and $[\![A]\!]_\Gamma : [\![\Gamma]\!] \to \mathrm{DOM}$ is a parametrization, Define

$$[\![\Gamma, x \in A]\!] = \Sigma([\![\Gamma]\!], [\![A]\!]_\Gamma)$$

which is an effective domain by Theorems 2.1 and 3.12. Note that if $\Gamma_i \equiv x_1 \in A_1, \ldots, x_{i-1} \in A_{i-1}$ for $i = 1, \ldots, n+1$ then

$$[\![\Gamma_{n+1}]\!] = \{\perp\} \, \Sigma \, [\![A_1]\!]_{\Gamma_1} \, \Sigma \, [\![A]\!]_{\Gamma_2} \Sigma \cdots \Sigma [\![A_n]\!]_{\Gamma_n}.$$

In the case we have a context $\Gamma$, $\Delta$ then we informally use the notation $(u, v) \in [\![\Gamma, \Delta]\!]$ where $u$ is the part of the tuple due to $\Gamma$ and $v$ is the part of the tuple due to $\Delta$. With this convention we now state the theorems about thinning and substitution.

**Theorem 5.1.** (i) *If* $\Gamma$, $\Delta$ **context** *and* $\Gamma$, $z \in D$, $\Delta$ **context** *then*

(a) $(u, v, w) \in [\![\Gamma, z \in D, \Delta]\!]$ *implies* $(u, w) \in [\![\Gamma, \Delta]\!]$, *and*

(b) $(u, w) \in [\![\Gamma, \Delta]\!]$ *and* $v \in [\![D]\!]_\Gamma(u)$ *implies* $(u, v, w) \in [\![\Gamma, z \in D, \Delta]\!]$.

(ii) *If* $\Gamma$, $\Delta \Rightarrow A$ **set** *and* $\Gamma$, $z \in D$, $\Delta \Rightarrow A$ **set** *and* $(u, v, w) \in [\![\Gamma, z \in D, \Delta]\!]$ *then*

$$[\![A]\!]_{\Gamma, z \in D, \Delta}(u, v, w) = [\![A]\!]_{\Gamma, \Delta}(u, w).$$

(iii) *If* $\Gamma$, $\Delta \Rightarrow a \in A$ *and* $\Gamma$, $z \in D$, $\Delta \Rightarrow a \in A$ *and* $(u, v, w) \in [\![\Gamma, z \in D, \Delta]\!]$ *then*

$$[\![a]\!]_{\Gamma, z \in D, \Delta}(u, v, w) = [\![a]\!]_{\Gamma, \Delta}(u, w).$$

**Remark.** The theorem is proved by induction on a derivation of $\Gamma$, $\Delta$ **context**, $\Gamma$, $\Delta \Rightarrow A$ **set** and $\Gamma$, $\Delta \Rightarrow a \in A$ respectively.

In the substitution theorem below, we use the notation $\boldsymbol{a} \equiv a_1, \ldots, a_n$; $\boldsymbol{x} \equiv x_1, \ldots, x_n$; $\boldsymbol{x} \in \boldsymbol{A} \equiv x_1 \in A_1, \ldots, x_n \in A_n$; $\Phi \equiv y_1 \in B_1, \ldots, y_m \in B_m$ and $\Phi_i \equiv y_1 \in B_1, \ldots, y_{i-1} \in B_{i-1}$.

**Theorem 5.2.** *Suppose*

$$\Gamma, \Delta \Rightarrow a_1 \in A_1, \ldots, \Gamma, \Delta \Rightarrow a_n \in A_n(a_1, \ldots, a_{n-1}/x_1, \ldots, x_{n-1})$$

*and* $\Gamma$, $\Delta$, $\boldsymbol{x} \in \boldsymbol{A}$, $\Phi$ **context**.

(i) *If* $\Gamma$, $\boldsymbol{x} \in \boldsymbol{A}$, $\Phi$ **context** *then*

$$[\![\Gamma, \Delta, \Phi(\boldsymbol{a}/\boldsymbol{x})]\!] = \{(u, v, w_1, \ldots, w_n) : (u, v) \in [\![\Gamma, \Delta]\!],$$
$$w_i \in [\![B_i]\!]_{\Gamma, \boldsymbol{x} \in \boldsymbol{A}, \Phi_i}(u, [\![a_1]\!]_{\Gamma, \Delta}(u, v), \ldots, [\![a_n]\!]_{\Gamma, \Delta}(u, v), w_1, \ldots, w_{i-1})\}.$$

(ii) *If* $\Gamma$, $\boldsymbol{x} \in \boldsymbol{A}$, $\Phi \Rightarrow D$ **set** *then*

$$[\![D(\boldsymbol{a}/\boldsymbol{x})]\!]_{\Gamma, \Delta, \Phi(\boldsymbol{a}/\boldsymbol{x})}(u, v, w)$$
$$= [\![D]\!]_{\Gamma, \boldsymbol{x} \in \boldsymbol{A}, \Phi}(u, [\![a_1]\!]_{\Gamma, \Delta}(u, v), \ldots, [\![a_n]\!]_{\Gamma, \Delta}(u, v), w).$$

(iii) *If* $\Gamma$, $\boldsymbol{x} \in \boldsymbol{A}$, $\Phi \Rightarrow d \in D$ *then*

$$[\![d(\boldsymbol{a}/\boldsymbol{x})]\!]_{\Gamma, \Delta, \Phi(\boldsymbol{a}/\boldsymbol{x})}(u, v, w)$$
$$= [\![d]\!]_{\Gamma, \boldsymbol{x} \in \boldsymbol{A}, \Phi}(u, [\![a_1]\!]_{\Gamma, \Delta}(u, v), \ldots, [\![a_n]\!]_{\Gamma, \Delta}(u, v), w).$$

**Remarks.** (i) The role of the assumption $\Gamma$, $\Delta$, $\boldsymbol{x} \in \boldsymbol{A}$, $\Phi$ **context** is only to make sure that the variables of $\Delta$ and $\Phi$ are not in conflict.

(ii) The theorem is proved by induction on a derivation of $\Gamma, \boldsymbol{x} \in \boldsymbol{A}, \ \Phi$ **context**, $\Gamma, \boldsymbol{x} \in \boldsymbol{A}, \ \Phi \Rightarrow D$ **set** and $\Gamma, \boldsymbol{x} \in \boldsymbol{A}, \ \Phi \Rightarrow d \in D$ respectively.

*The assumption rule*

By our general stipulation we have derivations of $\Gamma \equiv x_1 \in A_1, \ldots, x_n \in A_n$ **context**, $\Gamma \Rightarrow A_i$ **set** and $\Gamma_i \Rightarrow A_i$ **set**, where $\Gamma_i \equiv x_1 \in A_1, \ldots, x_{i-1} \in A_{i-1}$. Inductively, we thus have domains $[\![\Gamma]\!]$ and $[\![\Gamma_i]\!]$ and parametrizations $[\![A_i]\!]_\Gamma$ and $[\![A_i]\!]_{\Gamma_i}$. Define

$$[\![x_i]\!]_\Gamma = \pi_{i+1}^{n+1}.$$

Then $[\![x_i]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![A_i]\!]_\Gamma)$ by Lemma 2.7, since $[\![A_i]\!]_\Gamma(u, v) = [\![A_i]\!]_{\Gamma_i}(u)$ for each $(u, v) \in [\![\Gamma]\!]$ where $u \in [\![\Gamma_i]\!]$, by Theorem 5.1. Furthermore $[\![x_i]\!]_\Gamma$ is clearly effective.

*The equality rules*

These rules are trivial, given our interpretation of $\in$ and $=$.

*The cartesian product of a family of sets*

$\Pi$-*formation.* Inductively, $[\![\Gamma]\!]$ is a domain, $[\![A]\!]_\Gamma : [\![\Gamma]\!] \to \mathrm{DOM}$ is a parametrization, and $[\![B]\!]_{\Gamma, x \in A}$ is a parametrization over $[\![A]\!]_\Gamma$. Define $[\![(\Pi x \in A)B]\!]_\Gamma : [\![\Gamma]\!] \to \mathrm{DOM}$ by

$$[\![(\Pi x \in A)B]\!]_\Gamma = \Pi([\![A]\!]_\Gamma, [\![B]\!]_{\Gamma, x \in A}).$$

Then $[\![(\Pi x \in A)B]\!]_\Gamma$ is a parametrization by Theorem 2.11 and effective by Theorem 3.21.

**Remark.** The equality rule for $\Pi$-formation is trivially satisfied by our choice of interpretation of $\in$ and $=$. The same is true for all equality rules in formation, introduction and elimination rules so these rules will be ignored in the sequel.

$\Pi$-*introduction.* In addition to the assumptions above, we have inductively

$$[\![b]\!]_{\Gamma, x \in A} \in \Pi([\![\Gamma, x \in A]\!], [\![B]\!]_{\Gamma, x \in A}) = \Pi(\Sigma([\![\Gamma]\!], [\![A]\!]_\Gamma), [\![B]\!]_{\Gamma, x \in A}).$$

Define

$$[\![(\lambda x)b]\!]_\Gamma = \mathrm{curry}([\![b]\!]_{\Gamma, x \in A}).$$

Then $[\![(\lambda x)b]\!]_\Gamma \in \Pi([\![\Gamma]\!], \Pi([\![A]\!]_\Gamma, [\![B]\!]_{\Gamma, x \in A})) = \Pi([\![\Gamma]\!], [\![(\Pi x \in A)B]\!]_\Gamma)$ and effective by the effective version of Theorem 2.22.

$\Pi$-*elimination.* By the induction hypothesis: $[\![a]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![A]\!]_\Gamma)$ and

$$[\![c]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![(\Pi x \in A)B]\!]_\Gamma) = \Pi([\![\Gamma]\!], \Pi([\![A]\!]_\Gamma, [\![B]\!]_{\Gamma, x \in A})).$$

Define $[\![\mathrm{Ap}(c, a)]\!]_\Gamma$ for $w \in [\![\Gamma]\!]$ by

$$[\![\mathrm{Ap}(c, a)]\!]_\Gamma(w) = [\![c]\!]_\Gamma(w)[\![a]\!]_\Gamma(w).$$

Thus $[\![\mathrm{Ap}(c, a)]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![B(a/x)]\!]_\Gamma)$ as in the example following Theorem 2.22, using Theorem 5.2, and $[\![\mathrm{Ap}(c, a)]\!]_\Gamma$ is effective.

*$\Pi$-equality.* We inductively have the assumptions above. For $w \in [\![\Gamma]\!]$, using Theorem 5.2,

$$\begin{aligned}[\![\mathrm{Ap}((\lambda x)b, a)]\!]_\Gamma(w) &= [\![(\lambda x)b]\!]_\Gamma(w)[\![a]\!]_\Gamma(w)\\ &= \mathrm{curry}([\![b]\!]_{\Gamma,x\in A})(w)[\![a]\!]_\Gamma(w)\\ &= [\![b]\!]_{\Gamma,x\in A}(w, [\![a]\!]_\Gamma(w)) = [\![b(a/x)]\!]_\Gamma(w).\end{aligned}$$

*The disjoint union of a family of sets*

*$\Sigma$-formation.* Inductively, $[\![\Gamma]\!]$ is a domain, $[\![A]\!]_\Gamma : [\![\Gamma]\!] \to \mathrm{DOM}$ is a parametrization, and $[\![B]\!]_{\Gamma,x\in A}$ is a parametrization over $[\![A]\!]_\Gamma$. Define $[\![(\Sigma x \in A)B]\!]_\Gamma : [\![\Gamma]\!] \to \mathrm{DOM}$ by

$$[\![(\Sigma x \in A)B]\!]_\Gamma = \Sigma([\![A]\!]_\Gamma, [\![B]\!]_{\Gamma,x\in A}).$$

Then $[\![(\Sigma x \in A)B]\!]_\Gamma$ is a parametrization by Theorem 2.3 and effective by Theorem
3.20.

*$\Sigma$-introduction.* In addition to the assumptions above we inductively have

$$[\![a]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![A]\!]_\Gamma) \quad \text{and}$$
$$[\![b]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![B(a/x)]\!]_\Gamma) = \Pi([\![\Gamma]\!], (w)[\![B]\!]_{\Gamma,x\in A}(w, [\![a]\!]_\Gamma(w))).$$

Define $[\![(a, b)]\!]_\Gamma$, for $w \in [\![\Gamma]\!]$, by

$$[\![(a, b)]\!]_\Gamma(w) = ([\![a]\!]_\Gamma(w), [\![b]\!]_\Gamma(w)).$$

By Lemma 2.18, $[\![(a, b)]\!]_\Gamma \in \Pi([\![\Gamma]\!], \Sigma([\![A]\!]_\Gamma, [\![B]\!]_{\Gamma,x\in A})) = \Pi([\![\Gamma]\!], [\![(\Sigma x \in A)B]\!]_\Gamma)$ and is effective. Note that it is essential that $\Gamma, x \in A \Rightarrow B$ **set** appears in the derivation in order to obtain $[\![B]\!]_{\Gamma,x\in A}$ and not just $[\![B(a/x)]\!]_\Gamma$. This is the case, since, by our general stipulation, $\Gamma \Rightarrow (\Sigma x \in A)B$ **set** and hence $\Gamma, x \in A \Rightarrow B$ **set**.

*$\Sigma$-elimination.* Inductively we assume $[\![c]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![(\Sigma x \in A)B]\!]_\Gamma)$ and

$$[\![d]\!]_{\Gamma,x\in A,y\in B} \in \Pi([\![\Gamma]\!]\,\Sigma\,[\![A]\!]_\Gamma\,\Sigma\,[\![B]\!]_{\Gamma,x\in A}, [\![C((x, y)/z)]\!]_{\Gamma,a\in A,y\in B}).$$

Furthermore $[\![C]\!]_{\Gamma,z\in(\Sigma x\in A)B}$ is a parametrization by the overall hypothesis. Define $[\![E(c, (x, y)d)]\!]_\Gamma$, for $w \in [\![\Gamma]\!]$, by

$$[\![E(c, (x, y)d)]\!]_\Gamma(w) = [\![d]\!]_{\Gamma,x\in A,y\in B} \circ \phi(w, [\![c]\!]_\Gamma(w))$$

where $\phi$ is the function defined in the proof of Proposition 2.4.

Let $h:[\![\Gamma]\!]\to\Sigma([\![\Gamma]\!], [\![(\Sigma x\in A)B]\!]_\Gamma)$ be defined by $h(w)=(w, [\![c]\!]_\Gamma(w))$. Then $h$ is continuous by Lemma 2.13 and hence $[\![E(c, (x, y)d)]\!]_\Gamma$ is p-continuous w.r.t. $[\![C((x, y)/z)]\!]_{\Gamma,x\in A,y\in B}\circ\phi\circ h$. Let $w\in[\![\Gamma]\!]$ and $[\![c]\!]_\Gamma(w)=(u, v)$. Then

$$\begin{aligned}
[\![C((x, y)/z]\!]_{\Gamma,x\in A,y\in B}\circ\phi\circ h(w) &= [\![C((x, y)/z)]\!]_{\Gamma,x\in A,y\in B}(w, u, v)\\
&= [\![C]\!]_{\Gamma,z\in(\Sigma x\in A)B}(w, [\![(x, y)]\!]_{\Gamma,x\in A,y\in B}(w, u, v))\\
&= [\![C]\!]_{\Gamma,z\in(\Sigma x\in A)B}(w, ([\![x]\!]_{\Gamma,x\in A,y\in B}(w, u, v), [\![y]\!]_{\Gamma,x\in A,y\in B}(w, u, v)))\\
&= [\![C]\!]_{\Gamma,z\in(\Sigma x\in A)B}(w, (u, v))\\
&= [\![C]\!]_{\Gamma,z\in(\Sigma x\in A)B}(w, [\![c]\!]_\Gamma(w))\\
&= [\![C(c/z)]\!]_\Gamma(w).
\end{aligned}$$

Clearly $[\![E(c, (x, y)d)]\!]_\Gamma$ is effective by the effective versions of Lemma 2.13 and Proposition 2.4.

Let us pause to consider the use of Theorem 5.2 in the previous calculation. First of all, it is easy to show $\Gamma, x\in A,\ y\in B,\ z\in(\Sigma x\in A)$ **context** and $\Gamma, x\in A, y\in B\Rightarrow(x, y)\in(\Sigma x\in A)B$. Furthermore, the overall hypothesis $\Gamma, z\in(\Sigma x\in A)B\Rightarrow C$ **set** must appear in the derivation previous to the application of the $\Sigma$-rule. Thus Theorem 5.2 has already been proved for $\Gamma, z\in(\Sigma x\in A)B\Rightarrow C$ **set** (cf. the remark after Theorem 5.2) which makes the application of Theorem 5.2 valid. However the overall hypothesis $\Gamma, z\in(\Sigma x\in A)B\Rightarrow C$ **set** in the $\Sigma$-elimination rule is necessary for this to make sense, which reflects the necessity of the assumption in the formal rule. Analogous remarks hold for the elimination and equality rules for $+$, the $I$-type, $N_n$, $N$ and $\Omega$. In the sequel we leave the verification of the validity of our use of Theorems 5.1 and 5.2 to the reader.

*$\Sigma$-equality.* In addition to the assumptions above we also assume inductively that $[\![a]\!]_\Gamma\in\Pi([\![\Gamma]\!], [\![A]\!]_\Gamma)$ and $[\![b]\!]_\Gamma\in\Pi([\![\Gamma]\!], [\![B(a/x)]\!]_\Gamma)$. Then, for $w\in[\![\Gamma]\!]$,

$$\begin{aligned}
[\![E((a, b), (x, y)d]\!]_\Gamma(w) &= [\![d]\!]_{\Gamma,x\in A,y\in B}\circ\phi(w, [\![(a, b)]\!]_\Gamma(w))\\
&= [\![d]\!]_{\Gamma,x\in A,y\in B}\circ\phi(w, ([\![a]\!]_\Gamma(w), [\![b]\!]_\Gamma(w)))\\
&= [\![d]\!]_{\Gamma,x\in A,y\in B}(w, [\![a]\!]_\Gamma(w), [\![b]\!]_\Gamma(w))\\
&= [\![d(a, b/x, y)]\!]_\Gamma(w).
\end{aligned}$$

*The disjoint union of two sets*

*$+$-formation.* Inductively, $[\![\Gamma]\!]$ is a domain, $[\![A]\!]_\Gamma:[\![\Gamma]\!]\to\mathrm{DOM}$ is a parametrization, and $[\![B]\!]_\Gamma:[\![\Gamma]\!]\to\mathrm{DOM}$ is a parametrization. Define $[\![A+B]\!]_\Gamma:[\![\Gamma]\!]\to\mathrm{DOM}$ by

$$[\![A+B]\!]_\Gamma(w)=[\![A]\!]_\Gamma(w)+[\![B]\!]_\Gamma(w).$$

Then $[\![A+B]\!]_\Gamma$ is a parmetrization by Proposition 1.12 and it is clearly effective.

*+-introduction.* In addition to the assumptions above we also assume inductively that $[\![a]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![A]\!]_\Gamma)$ and $[\![b]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![B]\!]_\Gamma)$. Define $[\![i(a)]\!]_\Gamma$, for $w \in [\![\Gamma]\!]$, by

$$[\![i(a)]\!]_\Gamma(w) = (0, [\![a]\!]_\Gamma(w)) \quad \text{and} \quad [\![j(b)]\!]_\Gamma(w) = (1, [\![b]\!]_\Gamma(w)).$$

Then $[\![i(a)]\!]_\Gamma$ and $[\![j(b)]\!]_\Gamma$ are trivially p-continuous and effective w.r.t. $[\![A+B]\!]_\Gamma$.

*+-elimination.* Inductively we assume

$$\begin{aligned}
&[\![c]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![A+B]\!]_\Gamma),\\
&[\![d]\!]_{\Gamma, x\in A} \in \Pi(\Sigma([\![\Gamma]\!], [\![A]\!]_\Gamma), [\![C(i(x)/z)]\!]_{\Gamma, x\in A}) \quad \text{and}\\
&[\![e]\!]_{\Gamma, y\in B} \in \Pi(\Sigma([\![\Gamma]\!], [\![B]\!]_\Gamma), [\![C(j(y)/z)]\!]_{\Gamma, y\in B}).
\end{aligned}$$

By the overall hypothesis, $[\![C]\!]_{\Gamma, z\in A+B}$ is a parametrization over $\Sigma([\![\Gamma]\!], [\![A+B]\!]_\Gamma)$. For $w \in [\![\Gamma]\!]$, define

$$\begin{aligned}
[\![D(c, (x)d, (y)e)]\!]_\Gamma(w) &= \bot_{[\![C(c/z)]\!]_\Gamma(w)} && \text{if } [\![c]\!]_\Gamma(w) = \bot_{[\![A+B]\!]_\Gamma(w)},\\
&= [\![d]\!]_{\Gamma, x\in A}(w, u) && \text{if } [\![c]\!]_\Gamma(w) = (0, u),\\
&= [\![e]\!]_{\Gamma, y\in B}(w, v) && \text{if } [\![c]\!]_\Gamma(w) = (1, v).
\end{aligned}$$

First we show that $[\![D(c, (x)d, (y)e)]\!]_\Gamma(w) \in [\![C(c/z)]\!]_\Gamma(w)$ for each $w \in [\![\Gamma]\!]$. Suppose, for the typical nontrivial case that $[\![c]\!]_\Gamma(w) = (0, u)$. Then $u \in [\![A]\!]_\Gamma(w)$ and

$$\begin{aligned}
[\![D(c, (x)d, (y)e)]\!]_\Gamma(w) = [\![d]\!]_{\Gamma, x\in A}(w, u) &\in [\![C(i(x)/z)]\!]_{\Gamma, x\in A}(w, u)\\
&= [\![C]\!]_{\Gamma, z\in A+B}(w, [\![i(x)]\!]_{\Gamma, x\in A}(w, u))\\
&= [\![C]\!]_{\Gamma, z\in A+B}(w, (0, [\![x]\!]_{\Gamma, x\in A}(w, u)))\\
&= [\![C]\!]_{\Gamma, z\in A+B}(w, (0, u))\\
&= [\![C]\!]_{\Gamma, z\in A+B}(w, [\![c]\!]_\Gamma(w))\\
&= [\![C(c/z)]\!]_\Gamma(w).
\end{aligned}$$

A similar calculation shows that when $w \sqsubseteq v$ and $[\![c]\!]_\Gamma(w) = (0, w')$ say, and hence $[\![c]\!]_\Gamma(v) = (0, v')$ then

$$\begin{aligned}
&(w, w') \sqsubseteq (v, v') \quad \text{and}\\
&[\![C(c/z)]\!]_\Gamma^+[w, v] = [\![C(i(x)/z)]\!]_{\Gamma, x\in A}^+[(w, w'), (v, v')].
\end{aligned}$$

It remains to show that $[\![D(c, (x)d, (y)e)]\!]_\Gamma$ is p-continuous w.r.t. $[\![C(c/z)]\!]_\Gamma$. We leave the p-monotonicity to the reader. Assume

$$r \in \text{approx}_{[\![C(c/z)]\!]_\Gamma(w)}([\![D(c, (x)d, (y)e)]\!]_\Gamma(w)).$$

For the typical nontrivial case assume $[\![c]\!]_\Gamma(w) = (0, u)$ so that

$$[\![D(c, (x)d, (y)e)]\!]_\Gamma(w) = [\![d]\!]_{\Gamma, x\in A}(w, u) \in [\![C(i(x)/z)]\!]_{\Gamma, x\in A}(w, u).$$

By the continuity of $[\![d]\!]_{\Gamma,x\in A}$ there is $(a', b') \in \Sigma([\![\Gamma]\!], [\![A]\!]_\Gamma)_c$ such that $(a', b') \sqsubseteq (w, u)$ and $r \sqsubseteq [\![d]\!]_{\Gamma,x\in A}(a', b')^{(w,u)}$. Let $b'' = b'^{(w)}$ and consider $(0, b'') \in [\![A+B]\!]_\Gamma(w)_c$. Then by the p-continuity of $[\![c]\!]_\Gamma$ there is $a'' \in \mathrm{approx}(w)$ such that $(0, b'') \sqsubseteq [\![c]\!]_\Gamma(a'')^{(w)}$. Let $a = a' \sqcup a''$. Then $(0, b'') \sqsubseteq [\![c]\!]_\Gamma(a)^{(w)}$. It follows $[\![c]\!]_\Gamma(a) = (0, v)$ where $v \in [\![A]\!]_\Gamma(a)$ and $b'' \sqsubseteq v^{(w)}$. Let $b = b'^{(a)}$ so that $b^{(w)} = b''$. Then

$$(a, b) \in \Sigma([\![\Gamma]\!], [\![A]\!]_\Gamma)_c \quad \text{and} \quad (a, b) \sqsubseteq (w, u).$$

Furthermore, since $(a', b') \sqsubseteq (a, b)$,

$$r \sqsubseteq [\![d]\!]_{\Gamma,x\in A}(a, b)^{(w,u)}.$$

But $(a, b) \sqsubseteq (a, v)$ and hence

$$\begin{aligned} r &\sqsubseteq [\![d]\!]_{\Gamma,x\in A}(a, v)^{(w,u)} \\ &= [\![D(c, (x)d, (y)e)]\!]_\Gamma(a)^{(w,u)} \\ &= [\![D(c, (x)d, (y)e)]\!]_\Gamma(a)^{(w)} \in [\![C(c/z)]\!]_\Gamma(w). \end{aligned}$$

To see that $[\![D(c, (x)d, (y)e)]\!]_\Gamma$ is effective we have for $(t, r) \in \Sigma([\![\Gamma]\!], [\![C(c/z)]\!]_\Gamma)_c$,

$$\begin{aligned} t \sqsubseteq [\![D(c, (x)d, (y)e)]\!]_\Gamma(r) \quad &\Leftrightarrow \quad t = \bot_{[\![C(c/z)]\!]_\Gamma(r)} \\ &\text{or } (\exists s \in [\![A]\!]_\Gamma(r)_c)((0, s) \sqsubseteq [\![c]\!]_\Gamma(r) \;\&\; t \sqsubseteq [\![d]\!]_{\Gamma,x\in A}(r, s)) \\ &\text{or } (\exists s \in [\![B]\!]_\Gamma(r)_c)((1, s) \sqsubseteq [\![c]\!]_\Gamma(r) \;\&\; t \sqsubseteq [\![e]\!]_{\Gamma,y\in B}(r, s)) \end{aligned}$$

by the p-continuity of $[\![d]\!]_{\Gamma,x\in A}$ and $[\![e]\!]_{\Gamma,y\in B}$. The right-hand side is semidecidable so $[\![D(c, (x)d, (y)e)]\!]_\Gamma$ is indeed effective.

*+-equality.* We consider the first equality rule, the second being analogous. In addition to the assumptions for +-elimination we inductively have $[\![a]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![A]\!]_\Gamma)$ and $[\![i(a)]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![A+B]\!]_\Gamma)$. Then for $w \in [\![\Gamma]\!]$, $[\![i(a)]\!]_\Gamma(w) = (0, [\![a]\!]_\Gamma(w))$ and hence

$$[\![D(i(a), (x)d, (y)e)]\!]_\Gamma(w) = [\![d]\!]_{\Gamma,x\in A}(w, [\![a]\!]_\Gamma(w)) = [\![d(a/x)]\!]_\Gamma(w).$$

*The identity relation*

*I-formation.* Inductively, $[\![\Gamma]\!]$ is a domain, $[\![A]\!]_\Gamma : [\![\Gamma]\!] \to \mathrm{DOM}$ is a parametrization, $[\![a]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![A]\!]_\Gamma)$ and $[\![b]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![A]\!]_\Gamma)$. Then define $[\![I(A, a, b)]\!]_\Gamma$ by

$$[\![I(A, a, b)]\!]_\Gamma = I([\![A]\!]_\Gamma, [\![a]\!]_\Gamma, [\![b]\!]_\Gamma).$$

Thus $[\![I(A, a, b)]\!]_\Gamma$ is an effective parametrization over $[\![\Gamma]\!]$ by the effective version of Theorem 2.21.

*I-introduction.* Inductively, $[\![\Gamma]\!]$ is a domain, $[\![A]\!]_\Gamma : [\![\Gamma]\!] \to \mathrm{DOM}$ is a parametrization, and $[\![a]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![A]\!]_\Gamma)$. Define, for $w \in [\![\Gamma]\!]$,

$$[\![r(a)]\!]_\Gamma(w) = [\![a]\!]_\Gamma(w).$$

It is routine, unfolding the definitions, to verify that $[\![r(a)]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![I(A, a, a)]\!]_\Gamma)$ and that $[\![r(a)]\!]_\Gamma$ is effective.

*I-elimination.* In addition to the assumptions above, we inductively have that

$$[\![c]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![I(A, a, b)]\!]_\Gamma),$$

$$[\![d]\!]_{\Gamma, x\in A} \in \Pi(\Sigma([\![\Gamma]\!], [\![A]\!]_\Gamma), [\![C(x, r(x)/y, z)]\!]_{\Gamma, x\in A})$$

and that $[\![C]\!]_{\Gamma, x\in A, y\in A, z\in I(A,x,y)}$ is a parametrization over

$$[\![\Gamma]\!]\,\Sigma[\![A]\!]_\Gamma \Sigma[\![A]\!]_{\Gamma,x\in A}\, \Sigma[\![I(A, x, y)]\!]_{\Gamma, x\in A, y\in A}.$$

For ease of notation denote $[\![C]\!]_{\Gamma, x\in A, y\in A, z\in I(A,x,y)}$ by $F$. Now, $\Gamma, x \in A$ **context** so $\Gamma, x\in A \Rightarrow x \in A$ and $\Gamma, x\in A \Rightarrow r(x) \in I(A, x, x)$. Thus, by Theorem 5.2, for $(w, u) \in [\![\Gamma, x\in A]\!] = \Sigma([\![\Gamma]\!], [\![A]\!]_\Gamma)$,

$$\begin{aligned}[\![C(x, r(x)/y, z)]\!]_{\Gamma, x\in A}(w, u) &= F(w, u, [\![x]\!]_{\Gamma,x\in A}(w, u), [\![r(x)]\!]_{\Gamma,x\in A}(w, u)) \\ &= F(w, u, u, u).\end{aligned}$$

By the remark following Theorem 2.21, $[\![c]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![A]\!]_\Gamma)$. Thus

$$\begin{aligned}(w)[\![d]\!]_{\Gamma,x\in A}(w, [\![c]\!]_\Gamma(w)) &\in \Pi([\![\Gamma]\!], (w)[\![C(x, r(x)/y, z]\!]_{\Gamma,x\in A}(w, [\![c]\!]_\Gamma(w))) \\ &= \Pi([\![\Gamma]\!], (w)F(w, [\![c]\!]_\Gamma(w), [\![c]\!]_\Gamma(w), [\![c]\!]_\Gamma(w)).\end{aligned}$$

Define, for $w \in [\![\Gamma]\!]$, $[\![J(c, (x)d)]\!]_\Gamma(w) =$

$$F^+[(w, [\![c]\!]_\Gamma(w), [\![c]\!]_\Gamma(w), [\![c]\!]_\Gamma(w)), (w, [\![a]\!]_\Gamma(w), [\![b]\!]_\Gamma(w), [\![c]\!]_\Gamma(w))]\\([\![d]\!]_{\Gamma,x\in A}(w, [\![c]\!]_\Gamma(w))).$$

Clearly the right-hand side is well-defined, since $[\![c]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![I(A, a, b)]\!]_\Gamma)$ and $[\![J(c, (x)d)]\!]_\Gamma$ is p-continuous w.r.t. $(w)F(w, [\![a]\!]_\Gamma(w), [\![b]\!]_\Gamma(w), [\![c]\!]_\Gamma(w)) = [\![C(a, b, c/x, y, z)]\!]_\Gamma$ by Proposition 2.9 and Lemma 2.15. It is effective, using the effective versions of these results and the closure of effectivity under substitutions.

*I-equality.* Under the assumptions and notation above, for $w \in [\![\Gamma]\!]$,

$$\begin{aligned}&[\![J(r(a), (x)d]\!]_\Gamma(w) \\ &= F^+[(w, [\![r(a)]\!]_\Gamma(w), [\![r(a)]\!]_\Gamma(w), [\![r(a)]\!]_\Gamma(w)), (w, [\![a]\!]_\Gamma(w), [\![a]\!]_\Gamma(w), [\![r(a)]\!]_\Gamma(w))] \\ &\qquad ([\![d]\!]_{\Gamma,x\in A}(w, [\![r(a)]\!]_\Gamma(w))) \\ &= F^+[(w, [\![a]\!]_\Gamma(w), [\![a]\!]_\Gamma(w), [\![r(a)]\!]_\Gamma(w)), (w, [\![a]\!]_\Gamma(w), [\![a]\!]_\Gamma(w), [\![r(a)]\!]_\Gamma(w))] \\ &\qquad ([\![d]\!]_{\Gamma,x\in A}(w, [\![a]\!]_\Gamma(w))) \\ &= [\![d]\!]_{\Gamma,x\in A}(w, [\![a]\!]_\Gamma(w)) = [\![d(a/x)]\!]_\Gamma(w).\end{aligned}$$

**Remark.** The interpretation of the $I$-type is symmetric in the sense that $[\![I(A, a, b)]\!] = [\![I(A, b, a)]\!]$. However, it is not true that $\Gamma \Rightarrow I(A, a, b) = I(A, b, a)$

in general. To reflect the non-symmetric nature of the $I$-type one could define the interpretation by

$$[\![I(A, a, b)]\!]_\Gamma(w) = [\![A]\!]_\Gamma(w)^{[\![a]\!]_\Gamma(w)} \times [\![A]\!]_\Gamma(w)^{[\![b]\!]_\Gamma(w)}$$

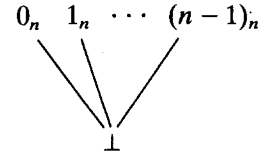
and modify the interpretations $[\![r(a)]\!]_\Gamma$ and $[\![J(c, (x)d)]\!]_\Gamma$ accordingly. This is a variant of a suggestion by I. Lindström.

*The finite sets*

*$N_n$-formation.* $[\![N_n]\!]_\Gamma : [\![\Gamma]\!] \to \mathrm{DOM}$ is defined by the constant parametrization

$$[\![N_n]\!]_\Gamma(w) = N_n,$$

where $N_n$ is the flat domain over $n$ elements:




Clearly, $[\![N_n]\!]_\Gamma$ is effective since $N_n$ is an effective domain.

*$N_n$-introduction.* For $m = 0, \ldots, n-1$ and $w \in [\![\Gamma]\!]$ define $[\![m_n]\!]_\Gamma(w) = m_n$. Then clearly $[\![m_n]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![N_n]\!]_\Gamma)$ and $[\![m_n]\!]_\Gamma$ is effective.

*$N_n$-elimination.* Inductively, we assume $[\![C]\!]_{\Gamma, z\in N_n}$ is a parametrization over $\Sigma([\![\Gamma]\!], [\![N_n]\!]_\Gamma) = [\![\Gamma]\!] \times N_n$, $[\![c]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![N_n]\!]_\Gamma)$ and $[\![c_m]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![C(m_n/z)]\!]_\Gamma)$ for $m = 0, \ldots, n-1$. Define, for $w \in [\![\Gamma]\!]$,

$$\begin{aligned}[\![R_n(c, c_0, \ldots, c_{n-1})]\!]_\Gamma(w) &= \perp_{[\![C(c/z)]\!]_\Gamma(w)} &&\text{if } [\![c]\!]_\Gamma(w) = \perp_{N_n},\\ &= [\![c_m]\!]_\Gamma(w) &&\text{if } [\![c]\!]_\Gamma(w) = m_n.\end{aligned}$$

Then, for $[\![c]\!]_\Gamma(w) = m_n$,

$$\begin{aligned}[\![R_n(c, c_0, \ldots, c_{n-1})]\!]_\Gamma(w) &\in [\![C(m_n/z)]\!]_\Gamma(w)\\ &= [\![C]\!]_{\Gamma, z\in N_n}(w, [\![m_n]\!]_\Gamma(w)) = [\![C]\!]_{\Gamma, z\in N_n}(w, m_n)\\ &= [\![C]\!]_{\Gamma, z\in N_n}(w, [\![c]\!]_\Gamma(w)) = [\![C(c/z)]\!]_\Gamma(w).\end{aligned}$$

It remains to show that $[\![R_n(c, c_0, \ldots, c_{n-1})]\!]_\Gamma$ is p-continuous w.r.t. $[\![C(c/z)]\!]_\Gamma$. We leave the p-monotonicity to the reader and assume $d \sqsubseteq [\![R_n(c, c_0, \ldots, c_{n-1})]\!]_\Gamma(w)$ where $d \in [\![C(c/z)]\!]_\Gamma(w)_c = [\![C]\!]_{\Gamma, z\in N_n}(w, [\![c]\!]_\Gamma(w))_c$. In case $[\![c]\!]_\Gamma(w) = \perp_{N_n}$ then $d = [\![R_n(c, c_0, \ldots, c_{n-1})]\!]_\Gamma(w) = \perp_{[\![C(c/z)_\Gamma(w)]\!]}$ so

$$[\![C(c/z)]\!]_\Gamma^+[\perp, w]([\![R_n(c, c_0, \ldots, c_{n-1})]\!]_\Gamma(\perp)) = d.$$

Now assume $[\![c]\!]_\Gamma(w) = m_n$. By the p-continuity of $[\![c]\!]_\Gamma$, choose $a' \in \mathrm{approx}(w)$ such that $[\![c]\!]_\Gamma(a') = m_n$. By the p-continuity of $[\![c_m]\!]_\Gamma$ choose $a'' \in \mathrm{approx}(w)$ such that

$$d \sqsubseteq [\![C(m_n/z)]\!]^+_\Gamma[a'', w]([\![c_m]\!]_\Gamma(a'')).$$

Let $a = a' \sqcup a''$. Then $[\![c]\!]_\Gamma(a) = m_n$ and

$$d \sqsubseteq [\![C(m_n/z)]\!]^+_\Gamma[a, w]([\![c_m]\!]_\Gamma(a)).$$

But

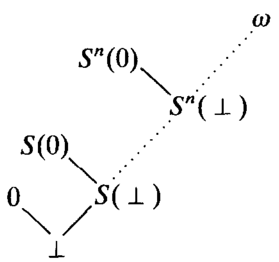
$$\begin{aligned}[\![C(m_n/z)]\!]^+_\Gamma[a, w] &= [\![C]\!]^+_{\Gamma, z\in N_n}[(a, [\![m_n]\!]_\Gamma(a)), (w, [\![m_n]\!]_\Gamma(w))] \\ &= [\![C]\!]^+_{\Gamma, z\in N_n}[(a, m_n), (w, m_n)] \\ &= [\![C]\!]^+_{\Gamma, z\in N_n}[(a, [\![c]\!]_\Gamma(a)), (w, [\![c]\!]_\Gamma(w))] \\ &= [\![C(c/z)]\!]^+_\Gamma[a, w].\end{aligned}$$

The effectiveness of $[\![R_n(c, c_0, \ldots, c_{n-1})]\!]_\Gamma$ is established similarly to the case for $[\![D(c, (x)d, (y)e]\!]_\Gamma$.

*$N_n$-equality.* These rules are trivial since $[\![m_n]\!]_\Gamma(w) = m_n \in \boldsymbol{N}_n$.

*The set of natural numbers*

*N-formation.* Let $\boldsymbol{N}$ be the domain pictured by




Then define $[\![N]\!]_\Gamma: [\![\Gamma]\!] \to \mathrm{DOM}$ to be the constant parametrization $[\![N]\!]_\Gamma(w) = \boldsymbol{N}$. Clearly $\boldsymbol{N}$ is an effective domain with a standard numbering enabling us to compute $n$ and 0 or $n$ and $\perp$ from an index for $S^n(0)$ or $S^n(\perp)$.

*N-introduction.* Define $S: \boldsymbol{N} \to \boldsymbol{N}$ by

$$S(S^n(\perp)) = S^{n+1}(\perp), \qquad S(S^n(0)) = S^{n+1}(0), \qquad S(\omega) = \omega.$$

Clearly $S$ is continuous and effective. Inductively, we assume $[\![a]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![N]\!]_\Gamma) = [\![\Gamma]\!] \to \boldsymbol{N}$. Define for $w \in [\![\Gamma]\!]$,

$$[\![0]\!]_\Gamma(w) = 0 \quad \text{and} \quad [\![s(a)]\!]_\Gamma(w) = S([\![a]\!]_\Gamma(w)).$$

These functions are clearly continuous and effective under our inductive assumptions.

*N-elimination.* Inductively, we assume $[\![C]\!]_{\Gamma,z\in N}$ is a parametrization over $\Sigma([\![\Gamma]\!], [\![N]\!]_\Gamma) = [\![\Gamma]\!] \times \boldsymbol{N}$,

$$[\![c]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![N]\!]_\Gamma) = [\![\Gamma]\!] \to \boldsymbol{N},$$

$$[\![d]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![C(0/z)]\!]_\Gamma) \quad \text{and}$$

$$[\![e]\!]_{\Gamma,x\in N,y\in C(x/z)} \in$$
$$\Pi([\![\Gamma]\!] \Sigma [\![N]\!]_\Gamma \Sigma [\![C(x/z)]\!]_{\Gamma,x\in N}, [\![C(s(x)/z)]\!]_{\Gamma,x\in N,y\in C(x/z)}).$$

We make the following computations. For $(w, v) \in \Sigma([\![\Gamma]\!], [\![N]\!]_\Gamma) = [\![\Gamma]\!] \times \boldsymbol{N}$,

$$\begin{aligned}[\![C(x/z)]\!]_{\Gamma,x\in N}(w, v) &= [\![C]\!]_{\Gamma,z\in N}(w, [\![x]\!]_{\Gamma,x\in N}(w, v)) \\ &= [\![C]\!]_{\Gamma,z\in N}(w, v).\end{aligned}$$

For $(w, v, u) \in [\![\Gamma]\!] \Sigma [\![N]\!]_\Gamma \Sigma [\![C(x/z)]\!]_{\Gamma,x\in N}$,

$$\begin{aligned}[\![C(s(x)/z)]\!]_{\Gamma,x\in N,y\in C(x/z)}(w, v, u) &= [\![C]\!]_{\Gamma,z\in N}(w, [\![s(x)]\!]_{\Gamma,x\in N,y\in C(x/z)}(w, v, u)) \\ &= [\![C]\!]_{\Gamma,z\in N}(w, [\![s(x)]\!]_{\Gamma,z\in N}(w, v)) \\ &= [\![C]\!]_{\Gamma,z\in N}(w, S([\![x]\!]_{\Gamma,x\in N}(w, v))) \\ &= [\![C]\!]_{\Gamma,z\in N}(w, S(v)).\end{aligned}$$

Furthermore, for $w \in [\![\Gamma]\!]$,

$$[\![C(0/z)]\!]_\Gamma(w) = [\![C]\!]_{\Gamma,z\in N}(w, [\![0]\!]_\Gamma(w)) = [\![C]\!]_{\Gamma,z\in N}(w, 0)$$

and

$$[\![C(c/z)]\!]_\Gamma(w) = [\![C]\!]_{\Gamma,z\in N}(w, [\![c]\!]_\Gamma(w)).$$

For ease of notation we denote $[\![C]\!]_{\Gamma,z\in N}$ by $F$. Define $f_{d,e}$, for $(w, a) \in [\![\Gamma]\!] \times \boldsymbol{N}_c$, inductively on $n$ by,

$$f_{d,e}(w, \bot) = \bot_{F(w,\bot)},$$

$$f_{d,e}(w, 0) = [\![d]\!]_\Gamma(w),$$

$$f_{d,e}(w, S^{n+1}(\bot)) = [\![e]\!]_{\Gamma,x\in N,y\in C(x/z)}(w, S^n(\bot), f_{d,e}(w, S^n(\bot))),$$

$$f_{d,e}(w, S^{n+1}(0)) = [\![e]\!]_{\Gamma,x\in N,y\in C(x/z)}(w, S^n(0), f_{d,e}(w, S^n(0))).$$

We show that $f_{d,e}(w, a) \in F(w, a)$ for each $w \in [\![\Gamma]\!]$ and $a \in \boldsymbol{N}_c$. It is certainly true for $a = \bot$ and $a = 0$. Suppose $f_{d,e}(w, S^n(\bot)) \in F(w, S^n(\bot))$. Then

$$\begin{aligned}f_{d,e}(w, S^{n+1}(\bot)) &= [\![e]\!]_{\Gamma,x\in N,y\in C(x/z)}(w, S^n(\bot), f_{d,e}(w, S^n(\bot))) \\ &\in [\![C(s(x)/z)]\!]_{\Gamma,x\in N,y\in C(x/z)}(w, S^n(\bot), f_{d,e}(w, S^n(\bot))) \\ &= F(w, S^{n+1}(\bot)).\end{aligned}$$

Similarly, $f_{d,e}(w, S^n(0)) \in F(w, S^n(0))$.

To show that $f_{d,e}$ is p-continuous w.r.t. $F$ it suffices by Lemma 2.17 to show that $f_{d,e}$ is p-continuous in each coordinate. By a trivial induction on $n$ one proves that $(w)f_{d,e}(w, a)$ is p-continuous w.r.t. $(w)F(w, a)$ for each fixed $a \in \boldsymbol{N}_c$, using our assumptions. Now we fix $w \in [\![\Gamma]\!]$ to show that $(t)f_{d,e}(w, t)$ is p-monotone on $\boldsymbol{N}_c$ w.r.t. $(t)F(w, t)$. First we show

$$F^+[(w, S^n(\bot)), (w, S^n(0))](f_{d,e}(w, S^n(\bot))) \sqsubseteq f_{d,e}(w, S^n(0))$$

by induction on $n$. It is clearly true for $n = 0$ so assume it is true for $n$. Then

$$f_{d,e}(w, S^{n+1}(\bot)) = [\![e]\!]_{\Gamma, x \in N, y \in C(x/z)}(w, S^n(\bot), f_{d,e}(w, S^n(\bot)))$$

and

$$f_{d,e}(w, S^{n+1}(0)) = [\![e]\!]_{\Gamma, x \in N, y \in C(x/z)}(w, S^n(0), f_{d,e}(w, S^n(0))).$$

We have $(w, S^n(\bot)) \sqsubseteq (w, S^n(0))$, and, by the induction hypothesis,

$$F^+[(w, S^n(\bot)), (w, S^n(0))](f_{d,e}(w, S^n(\bot))) \sqsubseteq f_{d,e}(w, S^n(0)).$$

But

$$F^+[(w, S^n(\bot)), (w, S^n(0))] = [\![C(x/z)]\!]^+_{\Gamma, x \in N}[(w, S^n(\bot)), (w, S^n(0))],$$

so that

$$(w, S^n(\bot), f_{d,e}(w, S^n(\bot))) \sqsubseteq (w, S^n(0), f_{d,e}(w, S^n(0))).$$

It follows that

$$\begin{aligned}&[\![C(s(x)/z)]\!]^+_{\Gamma, x \in N, y \in C(x/z)}[(w, S^n(\bot), f_{d,e}(w, S^n(\bot))), (w, S^n(0), f_{d,e}(w, S^n(0)))]\\ &\quad ([\![e]\!]_{\Gamma, x \in N, y \in C(x/z)}(w, S^n(\bot), f_{d,e}(w, S^n(\bot))))\\ &\sqsubseteq [\![e]\!]_{\Gamma, x \in N, y \in C(x/z)}(w, S^n(0), f_{d,e}(w, S^n(0))).\end{aligned}$$

But this is exactly

$$F^+[(w, S^{n+1}(\bot)), (w, S^{n+1}(0))](f_{d,e}(w, S^{n+1}(\bot))) \sqsubseteq f_{d,e}(w, S^{n+1}(0)).$$

A similar calculation shows

$$F^+[(w, S^n(\bot)), (w, S^{n+1}(\bot))](f_{d,e}(w, S^n(\bot))) \sqsubseteq f_{d,e}(w, S^{n+1}(\bot)).$$

Then it follows from the functorial properties that $(t)f_{d,e}(w, t)$ is p-monotone on $\boldsymbol{N}_c$. Considering the unique p-continuous extension of $(t)f_{d,e}(w, t)$ to $\boldsymbol{N}$ we have shown that $f_{d,e}$ is p-continuous w.r.t. $F$ by Lemma 2.17.

For $w \in [\![\Gamma]\!]$, define

$$[\![R(c, d, (x, y)e)]\!]_\Gamma(w) = f_{d,e}(w, [\![c]\!]_\Gamma(w)).$$

Then $[\![R(c, d, (x, y)e)]\!]_\Gamma \in \Pi([\![\Gamma]\!], (w)F(w, [\![c]\!]_\Gamma(w)))$. But

$$F(w, [\![c]\!]_\Gamma(w)) = [\![C]\!]_{\Gamma, z \in N}(w, [\![c]\!]_\Gamma(w)) = [\![C(c/z)]\!]_\Gamma(w)$$

so $[\![R(c, d, (x, y)e)]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![C(c/z)]\!]_\Gamma)$.

To show the effectiveness of $[\![R(c, d, (x, y)e)]\!]_\Gamma$ it clearly suffices to show that $f_{d,e}$ is effective. For this we must show that the set

$$\{(a, b, c) \in \Sigma([\![\Gamma]\!] \times \boldsymbol{N}, F)_c : c \sqsubseteq f_{d,e}(a, b)\}$$

is semidecidable. By the second recursion theorem there is a semidecidable set $W \subseteq \Sigma([\![\Gamma]\!] \times N, F)_c$ satisfying

$$\begin{aligned}(a, b, c) \in W \quad \Leftrightarrow \quad & (b = \bot \,\&\, c = \bot_{F(a,\bot)}) \\ & \text{or } (b = 0 \,\&\, c \sqsubseteq [\![d]\!]_\Gamma(a)) \\ & \text{or } (b = S^{n+1}(\bot) \,\&\, (\exists c')((a, S^n(\bot), c') \in W \\ & \qquad \&\, c \sqsubseteq [\![e]\!]_{\Gamma, x\in N, y \in C(x/z)}(a, S^n(\bot), c'))) \\ & \text{or } (b = S^{n+1}(0) \,\&\, (\exists c')((a, S^n(0), c') \in W \\ & \qquad \&\, c \sqsubseteq [\![e]\!]_{\Gamma, x\in N, y \in C(x/z)}(a, S^n(0), c'))).\end{aligned}$$

By an easy induction on $n$, using the p-continuity of $[\![e]\!]_{\Gamma, x\in N, y\in C(x/z)}$, it follows that $(a, b, c) \in W \Leftrightarrow c \sqsubseteq f_{d,e}(a, b)$.

*N-equality*. We continue to use the above notation. Consider the first equality rule. For $w \in [\![\Gamma]\!]$,

$$[\![R(0, d, (x, y)e]\!]_\Gamma(w) = f_{d,e}(w, [\![0]\!]_\Gamma(w)) = f_{d,e}(w, 0) = [\![d]\!]_\Gamma(w).$$

For the second equality rule we assume $[\![a]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![N]\!]_\Gamma)$. Then, for $w \in [\![\Gamma]\!]$,

$$\begin{aligned}[\![R(s(a), d, (x, y)e)]\!]_\Gamma(w) &= f_{d,e}(w, [\![s(a)]\!]_\Gamma(w)) = f_{d,e}(w, S([\![a]\!]_\Gamma(w))) \\ &= [\![e]\!]_{\Gamma, x\in N, y\in C(x/z)}(w, [\![a]\!]_\Gamma(w), f_{d,e}(w, [\![a]\!]_\Gamma(w))) \\ &= [\![e]\!]_{\Gamma, x\in N, y\in C(x/z)}(w, [\![a]\!]_\Gamma(w), [\![R(a, d, (x, y)e)]\!]_\Gamma(w)) \\ &= [\![e(a, R(a, d, (x, y)e)/x, y)]\!]_\Gamma(w).\end{aligned}$$

*The iteration set*

*Ω-formation*. Let $\Omega$ be the ordinal $\omega + 1 = \omega \cup \{\omega\}$. Then $\Omega$ is an effective domain with the usual ordering. Define $[\![\Omega]\!]_\Gamma : [\![\Gamma]\!] \to \mathrm{DOM}$ to be the constant parametrization $[\![\Omega]\!]_\Gamma(w) = \Omega$.

*Ω-introduction*. Let $S : \Omega \to \Omega$ be the continuous extension of the successor function, i.e. $S(n) = n + 1$ and $S(\omega) = \omega$. Inductively, we assume $[\![a]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![\Omega]\!]_\Gamma)$. Define, for $w \in [\![\Gamma]\!]$,

$$[\![a']\!]_\Gamma(w) = S([\![a]\!]_\Gamma(w)).$$

Then $[\![a']\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![\Omega]\!]_\Gamma)$. Trivially, $S$ is effective and hence $[\![a']\!]_\Gamma$ is effective.

*Ω-elimination*. Inductively, we assume $[\![C]\!]_{\Gamma, z\in\Omega}$ is a parametrization over $\Sigma([\![\Gamma]\!], [\![\Omega]\!]_\Gamma) = [\![\Gamma]\!] \times \Omega$, $[\![c]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![\Omega]\!]_\Gamma) = [\![\Gamma]\!] \to \Omega$ and

$$[\![d]\!]_{\Gamma, x\in\Omega, y\in C(x/z)} \in \Pi([\![\Gamma]\!] \,\Sigma\, [\![\Omega]\!]_\Gamma \,\Sigma\, [\![C(x/z)]\!]_{\Gamma, x\in\Omega}, [\![C(x'/z)]\!]_{\Gamma, x\in\Omega, y\in C(x/z)}).$$

Define $f_d$, for $(w, a) \in [\![\Gamma]\!] \times \Omega_c$ by

$$\begin{aligned}&f_d(w, 0) = \bot_{[\![C]\!]_{\Gamma, z\in\Omega}(w,0)}, \\ &f_d(w, n + 1) = [\![d]\!]_{\Gamma, x\in\Omega, y\in C(x/z)}(w, n, f_d(w, n)).\end{aligned}$$

With a proof quite analogous to the one for $N$-elimination, the unique continuous extension of $f$ to $[\![\Gamma]\!] \times \Omega$ is p-continuous w.r.t. $[\![C]\!]_{\Gamma, z \in \Omega}$. For $w \in [\![\Gamma]\!]$, define

$$[\![R_\omega(c, (x, y)d)]\!]_\Gamma(w) = f_d(w, [\![c]\!]_\Gamma(w)).$$

Then $[\![R_\omega(c, (x, y)d)]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![C(c/z)]\!]_\Gamma)$. The proof that $[\![R_\omega(c, (x, y)d)]\!]_\Gamma$ is effective is analogous to the proof for $[\![R(c, d, (x, y)e)]\!]_\Gamma$.

*$\Omega$-equality.* We use the above notation. Then, for $w \in [\![\Gamma]\!]$,

$$\begin{aligned}[\![R_\omega(a', (x, y)d)]\!]_\Gamma(w) &= f_d(w, [\![a']\!]_\Gamma(w)) = f_d(w, S([\![a]\!]_\Gamma(w)))\\ &= [\![d]\!]_{\Gamma, x \in \Omega, y \in C(x/z)}(w, f_d(w, [\![a]\!]_\Gamma(w)))\\ &= [\![d]\!]_{\Gamma, x \in \Omega, y \in C(x/z)}(w, [\![R_\omega(a, (x, y)d)]\!]_\Gamma(w))\\ &= [\![d(a, R_\omega(a, (x, y)d)/x, y)]\!]_\Gamma(w).\end{aligned}$$

*Additional axioms.* Define, for $w \in [\![\Gamma]\!]$, $[\![\omega]\!]_\Gamma(w) = \omega$. Clearly $[\![\omega]\!]_\Gamma \in \Pi([\![\Gamma]\!], [\![\Omega]\!]_\Gamma)$ and $[\![\omega]\!]_\Gamma$ is effective. And, finally,

$$[\![\omega']\!]_\Gamma(w) = S([\![\omega]\!]_\Gamma(w)) = S(\omega) = \omega = [\![\omega]\!]_\Gamma(w).$$

This completes the definition of the interpretation and the proof that it satisfies all rules of our formal system of intuitionistic type theory, given Theorems 5.1, 5.2 and 5.3 proved below.

**Proof of Theorem 5.1.** The proof of (i), (ii) and (iii) is by induction on a derivation of $\Gamma, \Delta$ **context**, $\Gamma, \Delta \Rightarrow A$ **set** and $\Gamma, \Delta \Rightarrow a \in A$ respectively. We only illustrate with a few cases, the others being similar.

(i): Consider (a) and suppose $(u, v, w) \in [\![\Gamma, z \in D, \Delta]\!]$. If $\Delta \equiv \emptyset$ then there is nothing to prove. So suppose $\Delta \equiv \Delta', x \in A$ and write $w$ as $w_1, w_2$ so that $w_2 \in [\![A]\!]_{\Gamma, z \in D, \Delta'}(u, v, w_1)$ and $(u, v, w_1) \in [\![\Gamma, z \in D, \Delta']\!]$. Both $\Gamma, \Delta'$ **context** and $\Gamma, \Delta' \Rightarrow A$ **set** have shorter derivations, so by induction,

$$(u, w_1) \in [\![\Gamma, \Delta']\!] \quad \text{and} \quad w_2 \in [\![A]\!]_{\Gamma, \Delta'}(u, w_1) = [\![A]\!]_{\Gamma, z \in D, \Delta'}(u, v, w_1).$$

Thus $(u, w_1, w_2) \in [\![\Gamma, \Delta', x \in A]\!]$ which proves (a).

(ii): Consider the case of $\Pi$-formation. Suppose $\Gamma, \Delta \Rightarrow (\Pi x \in B)C$ **set** and $\Gamma, z \in D, \Delta \Rightarrow (\Pi x \in B)C$ **set**. Then $\Gamma, \Delta \Rightarrow B$ **set** and $\Gamma, \Delta, x \in B \Rightarrow C$ **set** with shorter derivations. Thus, for $(u, v, w) \in [\![\Gamma, z \in D, \Delta]\!]$, we have inductively,

$$[\![B]\!]_{\Gamma, \Delta}(u, w) = [\![B]\!]_{\Gamma, z \in D, \Delta}(u, v, w)$$

and

$$(t)[\![C]\!]_{\Gamma, \Delta, x \in B}(u, w, t) = (t)[\![C]\!]_{\Gamma, z \in D, \Delta, x \in B}(u, v, w, t).$$

Thus

$$\begin{aligned}[\![(\Pi x \in B)C]\!]_{\Gamma, \Delta}(u, w) &= \Pi([\![B]\!]_{\Gamma, \Delta}(u, w), (t)[\![C]\!]_{\Gamma, \Delta, x \in B}(u, w, t))\\ &= \Pi([\![B]\!]_{\Gamma, z \in D, \Delta}(u, v, w), (t)[\![C]\!]_{\Gamma, z \in D, \Delta, x \in B}(u, v, w, t))\\ &= [\![(\Pi x \in B)C]\!]_{\Gamma, z \in D, \Delta}(u, v, w).\end{aligned}$$

(iii): Consider the case of $N$-elimination. Suppose $\Gamma, \Delta \Rightarrow R(c, d, (x, y)e) \in C(c/z)$ and $\Gamma, s \in D, \Delta \Rightarrow R(c, d, (x, y)e) \in C(c/z)$. In the notation used when defining the interpretation of $N$-elimination, we inductively have that (i) holds for $F$ and (ii) holds for $d$ and $e$. Then it is easily seen, by induction on $n$, for $(u, v, w) \in [\![\Gamma, s \in D, \Delta]\!]$ and $a \in N_c$, that $f_{d,e}(u, v, w, a) = f_{d,e}(u, w, a)$. It then follows that

$$[\![R(c, d, (x, y)e)]\!]_{\Gamma,\Delta}(u, w) = [\![R(c, d, (x, y)e)]\!]_{\Gamma, s\in D, \Delta}(u, v, w). \qquad \square$$

**Proof of Theorem 5.2.** The proof of (i), (ii) and (iii) is by induction on a derivation of $\Gamma$, $\boldsymbol{x} \in \boldsymbol{A}$, $\Phi$ **context**, $\Gamma, \boldsymbol{x} \in \boldsymbol{A}$, $\Phi \Rightarrow D$ **set** and $\Gamma, \boldsymbol{x} \in \boldsymbol{A}$, $\Phi \Rightarrow d \in D$ respectively. We only do some illustrative cases.

(i): If $\Phi \equiv \emptyset$ there is nothing to prove. So suppose $\Phi \equiv \Phi'$, $y \in B$. Then $\Gamma, \boldsymbol{x} \in \boldsymbol{A}$, $\Phi' \Rightarrow B$ **set** with a shorter derivation, so

$$\begin{aligned} &[\![B(\boldsymbol{a}/\boldsymbol{x})]\!]_{\Gamma,\Delta,\Phi'(\boldsymbol{a}/\boldsymbol{x})}(u, v, w) \\ &\quad = [\![B]\!]_{\Gamma,\boldsymbol{x}\in\boldsymbol{A},\Phi'}(u, [\![a_1]\!]_{\Gamma,\Delta}(u, v), \ldots, [\![a_n]\!]_{\Gamma,\Delta}(u, v), w). \end{aligned}$$

Now (i) follows, since (i) holds for $[\![\Gamma, \Delta, \Phi'(\boldsymbol{a}/\boldsymbol{x})]\!]$ by the induction hypothesis.

(ii): Consider the $\Pi$-formation rule. So suppose $\Gamma, \boldsymbol{x} \in \boldsymbol{A}$, $\Phi \Rightarrow (\Pi z \in B)C$ **set**. Then

$$\begin{aligned} &[\![((\Pi z \in B)C)(\boldsymbol{a}/\boldsymbol{x})]\!]_{\Gamma,\Delta,\Phi(\boldsymbol{a}/\boldsymbol{x})}(u, v, w) \\ &\quad = [\![(\Pi z \in B(\boldsymbol{a}/\boldsymbol{x}))C(\boldsymbol{a}/\boldsymbol{x})]\!]_{\Gamma,\Delta,\Phi(\boldsymbol{a}/\boldsymbol{x})}(u, v, w) \\ &\quad = \Pi([\![B(\boldsymbol{a}/\boldsymbol{x})]\!]_{\Gamma,\Delta,\Phi(\boldsymbol{a}/\boldsymbol{x})}(u, v, w), (t)[\![C(\boldsymbol{a}/\boldsymbol{x})]\!]_{\Gamma,\Delta,\Phi'(\boldsymbol{a}/\boldsymbol{x}),z\in B(\boldsymbol{a}/\boldsymbol{x})}(u, v, w, t)) \\ &\quad = \Pi([\![B]\!]_{\Gamma,\boldsymbol{x}\in\boldsymbol{A},\Phi}(u, [\![a_1]\!]_{\Gamma,\Delta}(u, v), \ldots, [\![a_n]\!]_{\Gamma,\Delta}(u, v), w), \\ &\qquad\qquad (t)[\![C]\!]_{\Gamma,\boldsymbol{x}\in\boldsymbol{A},\Phi,z\in B}(u, [\![a_1]\!]_{\Gamma,\Delta}(u, v), \ldots, [\![a_n]\!]_{\Gamma,\Delta}(u, v), w, t)) \\ &\quad = [\![(\Pi z \in B)C]\!]_{\Gamma,\boldsymbol{x}\in\boldsymbol{A},\Phi}(u, [\![a_1]\!]_{\Gamma,\Delta}(u, v), \ldots, [\![a_n]\!]_{\Gamma,\Delta}(u, v), w) \end{aligned}$$

where the next to least equality uses the induction hypotheses and (i).

(iii): First consider the assumption rule $\Gamma, \boldsymbol{x} \in \boldsymbol{A}$, $\Phi \Rightarrow z \in D$. If $z \in D$ comes from $\Gamma$ or $\Phi$ then (iii) holds by the induction hypothesis on $D$. So suppose $z \in D \equiv x_1 \in A_i$. Then

$$\begin{aligned} [\![z(\boldsymbol{a}/\boldsymbol{x})]\!]_{\Gamma,\Delta,\Phi(\boldsymbol{a}/\boldsymbol{x})}(u, v, w) &= [\![a_i]\!]_{\Gamma,\Delta,\Phi(\boldsymbol{a}/\boldsymbol{x})}(u, v, w) = [\![a_i]\!]_{\Gamma,\Delta}(u, v) \\ &= [\![z]\!]_{\Gamma,\boldsymbol{x}\in\boldsymbol{A},\Phi}(u, [\![a_1]\!]_{\Gamma,\Delta}(u, v), \ldots, [\![a_n]\!]_{\Gamma,\Delta}(u, v), w) \end{aligned}$$

where the next to last equality is Theorem 5.1.

Finally consider the $N$-elimination rule. In the notation used when giving the interpretation, we inductively have that (ii) holds for $F$ and (iii) holds for $d$ and $e$. By induction on $n$, it is easily seen that (iii) holds for $f_{d,e}$, that is

$$f_{d(\boldsymbol{a}/\boldsymbol{x}),e(\boldsymbol{a}/\boldsymbol{x})}(u, v, w, a) = f_{d,e}(u, [\![a_1]\!]_{\Gamma,\Delta}(u, v), \ldots, [\![a_n]\!]_{\Gamma,\Delta}(u, v), w, a).$$

Then

$$\begin{aligned}
&[\![R(c, d, (x, y)e)(\boldsymbol{a}/\boldsymbol{x})]\!]_{\Gamma,\Delta,\Phi(\boldsymbol{a}/\boldsymbol{x})}(u, v, w)\\
&\quad = [\![R(c(\boldsymbol{a}/\boldsymbol{x}), d(\boldsymbol{a}/\boldsymbol{x}), (x, y)e(\boldsymbol{a}/\boldsymbol{x}))]\!]_{\Gamma,\Delta,\Phi(\boldsymbol{a}/\boldsymbol{x})}(u, v, w)\\
&\quad = f_{d(\boldsymbol{a}/\boldsymbol{x}),e(\boldsymbol{a}/\boldsymbol{x})}(u, v, w, [\![c]\!]_{\Gamma,\Delta,\Phi(\boldsymbol{a}/\boldsymbol{x})}(u, v, w))\\
&\quad = f_{d,e}(u, [\![a_1]\!]_{\Gamma,\Delta}(u, v), \ldots, [\![a_n]\!]_{\Gamma,\Delta}(u, v), w,\\
&\qquad\quad [\![c]\!]_{\Gamma,\boldsymbol{x}\in\boldsymbol{A},\Phi}(u, [\![a_1]\!]_{\Gamma,\Delta}(u, v), \ldots, [\![a_n]\!]_{\Gamma,\Delta}(u, v), w))\\
&\quad = [\![R(c, d, (x, y)e)]\!]_{\Gamma,\boldsymbol{x}\in\boldsymbol{A},\Phi}(u, [\![a_1]\!]_{\Gamma,\Delta}(u, v), \ldots, [\![a_n]\!]_{\Gamma,\Delta}(u, v), w). \quad \square
\end{aligned}$$

Finally we prove that our interpretation is well-defined and that interpretations of types and terms depend only on contexts. Recall that we deal with the monomorphic type theory.

**Theorem 5.3.** (i) *If $\Gamma$* **context** *then $[\![\Gamma]\!]$ is independent of its derivation.*

(ii) *If $\Gamma \Rightarrow A$* **set** *then $[\![A]\!]_\Gamma$ is independent of its derivation.*

(iii) *If $\Gamma \Rightarrow a \in A$ then $[\![a]\!]_\Gamma$ is independent of its derivation and of $A$.*

**Proof.** The proof of (i), (ii) and (iii) is by induction on a derivation of $\Gamma$ **context**, $\Gamma \Rightarrow A$ **set** and $\Gamma \Rightarrow a \in A$, respectively.

(i): Suppose $\Gamma$ **context**. If $\Gamma \equiv \emptyset$ then $[\![\Gamma]\!] = \{\bot\}$ and (i) holds by the nature of the context rule. Suppose $\Gamma \equiv \Gamma', x \in A$. Then, with shorter derivations, $\Gamma'$ **context** and $\Gamma' \Rightarrow A$ **set**. By induction $[\![\Gamma']\!]$ and $[\![A]\!]_{\Gamma'}$ are independent of derivations and hence $[\![\Gamma]\!] = \Sigma([\![\Gamma']\!], [\![A]\!]_{\Gamma'})$ is independent of derivation.

(ii): Similar to (i).

(iii): We exemplify with the +-elimination rule, the other introduction and elimination rules being similar. So suppose

$$\Gamma \Rightarrow D(A, B, (z)C, c, (x)d, (y)e) \in C(c/z)$$

The interpretation $[\![D(A, B, (z)C, c, (x)d, (y)e)]\!]_\Gamma$ is defined in terms of $[\![C]\!]_{\Gamma,z\in A+B}, [\![c]\!]_\Gamma$, $[\![d]\!]_{\Gamma,x\in A}$ and $[\![e]\!]_{\Gamma,y\in B}$, and this information is contained in the term. By induction, (ii) holds for $[\![C]\!]_{\Gamma,z\in A+B}$ and (iii) holds for $[\![c]\!]_\Gamma$, $[\![d]\!]_{\Gamma,x\in A}$ and $[\![e]\!]_{\Gamma,y\in B}$. Thus, whenever

$$D(A, B, (z)C, c, (x)d, (y)e)$$

has context $\Gamma$ and is obtained by +-elimination, its interpretation is the one given above. The only other way to derive

$$\Gamma \Rightarrow D(A, B, (z)C, c, (x)d, (y)e) \in E$$

is by the equality of sets rule. But the equality of sets rule can only be applied, in our case, after an application of the +-elimination rule, which fixes the interpretation. $\square$

## 6. Operational and denotational semantics

The domain interpretation given in the previous section is not adequate for the operational semantics of type theory, that is the denotations do not respect the way terms are computed. In this section we shall modify the domain interpretation, or denotational semantics, of Section 5 to obtain one which is adequate for, and in fact equivalent to, the operational semantics in a strong sense. The method of proof is analogous to the one given by Martin-Löf [8]. Plotkin [14] is also relevant here.

We now describe computation rules for closed terms of intuitionistic type theory, in other words we give its operational semantics. The given rules correspond to a lazy evaluation. A deterministic computation rule $a \to b$ is defined for each closed term $a$. Computations are then obtained by the transitive closure of this relation, also denoted by $\to$. A term is *canonical* if it is obtained by an introduction rule. Thus a term is canonical if it has the form $(\lambda x)b$, $(a, b)$, $i(a)$, $j(b)$, $r(a)$, $m_n$, 0, $s(a)$, or $a'$. If $a$ is a closed canonical term then $a \to a$, so that a closed canonical term has itself as value. The remaining computation rules are essentially given by the various equality rules, considering these as term rewriting rules, as follows. To compute $\mathrm{Ap}(c, a)$ first compute $c$. If this computation terminates, that is the computation gives a canonical term in finitely many steps, then we must have $c \to (\lambda x)b$ for some $b$. Then set $\mathrm{Ap}(c, a) \to b(a/x)$. Put more succinctly,

$$\mathrm{Ap}(c, a) \to b(a/x) \quad \text{if } c \to (\lambda x)b.$$

Similarly,

$$\begin{aligned}
&E(c, (x, y)d) \to d(a, b/x, y) \quad \text{if } c \to (a, b),\\
&D(c, (x)d, (y)e) \to d(a/x) \quad \text{if } c \to i(a),\\
&D(c, (x)d, (y)e) \to e(b/y) \quad \text{if } c \to j(b),\\
&J(c, (x)d) \to d(a/x) \quad \text{if } c \to r(a),\\
&R_n(c, c_0, \ldots, c_{n-1}) \to c_m \quad \text{if } c \to m_n,\\
&R(c, d, (x, y)e) \to d \quad \text{if } c \to 0,\\
&R(c, d, (x, y)e) \to e(a, R(a, d, (x, y)e)/x, y) \quad \text{if } c \to s(a),\\
&R_\omega(c, (x, y)d) \to d(a, R_\omega(a, (x, y)d)/x, y) \quad \text{if } c \to a', \quad \text{and}\\
&\omega \to \omega'.
\end{aligned}$$

The computation of a term *a terminates* if $a \to b$ where $b$ is a canonical terms, and $b$ is said to be *the value of* the term $a$. Not all computations terminate in partial type theory, in constrast to the case when the iteration type is not present. For example, the term $R_\omega(\omega, (x, y)y)$ does not terminate.

A minimal requirement for a denotational semantics to be adequate for an operational semantics is that a term $c$ terminates if and only if $[\![c]\!] \neq \bot$. For the denotational semantics given in Section 5 it is easily shown that $[\![(\lambda x)\mathrm{Ap}(c, x)]\!] = [\![c]\!]$, whenever $\emptyset \Rightarrow c \in (\Pi x \in A)B$, that is the $\eta$-rule holds in the interpretation. However, the term $(\lambda x)\mathrm{Ap}(c, x)$ terminates with itself as value since it is a canonical term, even though $c$ may be a nonterminating term, for example the one given above. Similarly, when $\emptyset \Rightarrow c \in (\Sigma x \in A)B$, if $p(c) \equiv E(c, (x, y)x)$ and $q(c) \equiv E(c, (x, y)y)$ are the defined projections then $[\![(p(c), q(c))]\!] = [\![c]\!]$ even though $(p(c), q(c))$ terminates whereas $c$ may not.

Now we modify the domain interpretation of Section 5 to obtain a denotational semantics adequate for the given operational semantics. We refer to Section 5 for the appropriate assumptions for our definitions. Recall that if $F: D \to \mathrm{DOM}$ is a continuous functor then the functor $G: D \to \mathrm{DOM}$ defined by $G(w) = F(w)_\perp$, the lift of $F(w)$, is continuous by Proposition 1.12. Furthermore, the embedding $l: E \to E_\perp$ defined by $l(x) = x$ is continuous.

*Π-formation.* Define $[\![(\Pi x \in A)B]\!]_\Gamma(w) = \Pi([\![A]\!]_\Gamma, [\![B]\!]_{\Gamma, x\in A})(w)_\perp$.

*Π-introduction.* Define $[\![(\lambda x)b]\!]_\Gamma = l \circ \mathrm{curry}([\![b]\!]_{\Gamma, x\in A})$.

*Π-elimination.* Define
$$\begin{aligned}[\![\mathrm{Ap}(c, a)]\!]_\Gamma(w) &= [\![c]\!]_\Gamma(w)([\![a]\!]_\Gamma(w)) && \text{if } [\![c]\!]_\Gamma(w) \neq \perp,\\ &= \perp_{[\![B(a/x)]\!]_\Gamma(w)} && \text{if } [\![c]\!]_\Gamma(w) = \perp.\end{aligned}$$

*Σ-formation.* Define $[\![(\Sigma x \in A)B]\!]_\Gamma(w) = \Sigma([\![A]\!]_\Gamma, [\![B]\!]_{\Gamma, x\in A})(w)_\perp$.

*Σ-introduction.* Define $[\![(a, b)]\!]_\Gamma(w) = l(([\![a]\!]_\Gamma(w), [\![b]\!]_\Gamma(w)))$.

*Σ-elimination.* Define
$$\begin{aligned}[\![E(c, (x, y)d)]\!]_\Gamma(w) &= [\![d]\!]_{\Gamma, x\in A, y\in B} \circ \phi(w, [\![c]\!]_\Gamma(w)) && \text{if } [\![c]\!]_\Gamma(w) \neq \perp,\\ &= \perp_{[\![C(c/z)]\!]_\Gamma(w)} && \text{if } [\![c]\!]_\Gamma(w) = \perp.\end{aligned}$$

*I-formation.* Define $[\![I(A, a, b)]\!]_\Gamma(w) = I([\![A]\!]_\Gamma, [\![a]\!]_\Gamma, [\![b]\!]_\Gamma)(w)_\perp$.

*I-introduction.* Define $[\![r(a)]\!]_\Gamma(w) = l([\![a]\!]_\Gamma(w))$.

*I-elimination.* Define $[\![J(c, (x)d)]\!]_\Gamma(w) =$
$$\begin{aligned}&F^+[(w, [\![c]\!]_\Gamma(w), [\![c]\!]_\Gamma(w), [\![c]\!]_\Gamma(w)), (w, [\![a]\!]_\Gamma(w), [\![b]\!]_\Gamma(w), [\![c]\!]_\Gamma(w))]\\ &\qquad ([\![d]\!]_{\Gamma, x\in A}(w, [\![c]\!]_\Gamma(w))) \quad \text{if } [\![c]\!]_\Gamma(w) \neq \perp,\\ &[\![J(c, (x)d)]\!]_\Gamma(w) = \perp_{[\![C(a,b,c/x,y,z)]\!]_\Gamma(w)} \quad \text{if } [\![c]\!]_\Gamma(w) = \perp.\end{aligned}$$

The remaining parts of the interpretation are left unchanged. Note that we have altered the interpretation as little as possible, just making sure that canonical terms are not interpreted by $\perp$. It is routine to alter the proof in Section 5, to show that this modified interpretation gives us effective continuous functors and effective p-continuous functions, and that all rules are satisfied under the interpretation. From now on, the modified interpretation is the denotational semantics that we consider.

In the remaining part of this section we use the following notational conventions. Letters $a, b, c, \ldots$ will vary over terms, i.e. expressions for elements of types, while $A, B, C, \ldots$ will vary over expressions for types. Recall that an expression $A$ is meaningful only if $\Gamma \Rightarrow A$ **set** is a judgement for some $\Gamma$, i.e. derived in the formal system of Section 4. Similarly, an expression $a$ is meaningful only if $\Gamma \Rightarrow a \in A$ is a judgement for some $\Gamma$ and $A$. We also let $u, v, w, \ldots$ vary over *compact* elements in appropriate domains. Finally we often write $[\![a]\!]$ and $[\![A]\!]$ for $[\![a]\!]_\emptyset(\perp)$ and $[\![A]\!]_\emptyset(\perp)$.

We shall now define what we mean by an operational approximation of a closed term $c$. Our main result is then that $w$ is an operational approximation of $c$ if and only if $w$ is a denotational approximation of $c$, i.e. $w \sqsubseteq [\![c]\!]$.

**Definition 6.1.** Suppose $\emptyset \Rightarrow c \in D$. Then $u \in [\![D]\!]_c$ is an *operational approximation* of $c$, denoted $u \prec c$, is defined inductively on the complexity of $D$ as follows:

(i) $\perp_{[D]} \prec c$.

(ii) $D \equiv N_n$. Then $m_n \prec c$ if $c \to m_n$.

(iii) $D \equiv N$. Then $0 \prec c$ if $c \to 0$ and $S(u) \prec c$ if $(\exists a)(c \to s(a)\ \&\ u \prec a)$.

(iv) $D \equiv \Omega$. Then $n+1 \prec c$ if $(\exists a)(c \to a'\ \&\ n \prec a)$.

(v) $D \equiv (\Sigma x \in A)B$. Then $(u, v) \prec c$ if $(\exists a, b)(c \to (a, b)\ \&\ u \prec a\ \&\ v^{([\![a]\!])} \prec b)$.

(vi) $D \equiv (\Pi x \in A)B$. Then $\bigsqcup_i \langle u_i; v_i \rangle \prec c$ if $(\exists b)(c \to (\lambda x)b\ \&\ (\forall a \in A)(\forall i)(u_i \prec a$ implies $v_i^{([\![a]\!])} \prec b(a/x)))$.

**Remark.** The $\forall$ quantifier ranges over all $a$ such that $\emptyset \Rightarrow a \in A$ is a judgement.

(vii) $D \equiv A + B$. Then $(0, u) \prec c$ if $(\exists a)(c \to i(a)\ \&\ u \prec a)$ and $(1, v) \prec c$ if $(\exists b)(c \to j(b)\ \&\ v \prec b)$.

(viii) $D \equiv I(A, a, b)$. Then $l(u) \prec c$ if $(\exists d)(c \to r(d)\ \&\ u \prec d)$.

The following observation, proved by induction on $D$, gives the trivial direction of our main result. Note that the result is necessary in order that the definition should make sense.

**Lemma 6.2.** *Suppose $\emptyset \Rightarrow c \in D$. If $u \prec c$ then $u \sqsubseteq [\![c]\!]$.*

Another important observation, which, tacitly, often will be used, is that if $a \to b$ and $u \prec b$ then also $u \prec a$.

**Lemma 6.3.** *Suppose $\emptyset \Rightarrow c \in D$. Then $c$ terminates if and only if there is $u \in [\![D]\!]_c$ such that $u \prec c$ and $u \neq \perp_{[D]}$.*

**Proof.** Suppose $c$ terminates. Then using the appropriate clause from among (ii)–(viii), depending on $D$, in combination with clause (i), we easily construct $u \in [\![D]\!]_c$ such that $u < c$ and $u \neq \perp_{[\![D]\!]}$. In case $c$ does not terminate then clause (i) is the only one to give operational approximations. $\square$

Once we have proved that operational and denotational approximations are identical we have, by Lemma 6.3, that $c$ terminates if and only if $[\![c]\!] \neq \perp$.

**Lemma 6.4.** *Suppose $\emptyset \Rightarrow c \in D$. If $u < c$ and $v < c$ and $u$ and $v$ are consistent then $u \sqcup v < c$.*

**Proof.** Of course we need only consider the case when both $u$ and $v$ differ from $\perp$. The proof is by induction on $D$. We consider some cases leaving the rest to the reader.

*Case* $D \equiv N$: Trivially true since $u$ and $v$ are consistent if and only if they are comparable.

*Case* $D \equiv (\Sigma x \in A)B$: Suppose $(u_1, v_1) < c$, $(u_2, v_2) < c$ and $(u_1, v_1)$ and $(u_2, v_2)$ are consistent. Recall that $(u_1, v_1) \sqcup (u_2, v_2) = (u_1 \sqcup u_2, v_1^{(u_1 \sqcup u_2)} \sqcup v_2^{(u_1 \sqcup u_2)})$. Since $(u_i, v_i) < c$, there is $a$, $b$ such that $c \to (a, b)$ and $u_i < a$ and $v_i^{([\![a]\!])} < b$. Inductively, $u_1 \sqcup u_2 < a$ and $v_1^{([\![a]\!])} \sqcup v_2^{([\![a]\!])} < b$. Projection embeddings preserve arbitrary suprema so that

$$\begin{aligned}(v_1^{(u_1 \sqcup u_2)} \sqcup v_2^{(u_1 \sqcup u_2)})^{([\![a]\!])} &= v_1^{(u_1 \sqcup u_2)([\![a]\!])} \sqcup v_2^{(u_1 \sqcup u_2)([\![a]\!])} \\ &= v_1^{([\![a]\!])} \sqcup v_2^{([\![a]\!])}.\end{aligned}$$

But then $(u_1, v_1) \sqcup (u_2, v_2) < c$ by clause (v). $\square$

**Lemma 6.5.** *Suppose $\emptyset \Rightarrow c \in D$. If $u < c$ and $v \sqsubseteq u$ then $v < c$.*

**Proof.** If $v = \perp$ then $v < c$ by clause (i). We therefore assume $v \neq \perp$ and perform the proof by induction on the complexity of $D$. We do some cases, leaving the rest for the reader.

*Case* $D \equiv N$: By induction on $n$, where $u = S^n(\perp)$ or $u = S^n(0)$. The case $n = 0$ is trivial. So suppose $u = S(u')$. Then $v = S(v')$ and $v' \sqsubseteq u'$. Since $u < c$, there is $a$ such that $c \to s(a)$ and $u' < a$. But then $v' < a$, inductively, and hence $v < c$.

*Case* $D \equiv (\Sigma x \in A)B$: Suppose $(v_1, v_2) \sqsubseteq (u_1, u_2) < c$. Thus $v_1 \sqsubseteq u_1$ and $v_2^{(u_1)} \sqsubseteq u_2$. Furthermore there is $a$, $b$ such that $c \to (a, b)$ and $u_1 < a$ and $u_2^{([\![a]\!])} < b$. Note that

$$v_2^{([\![a]\!])} = v_2^{(u_1)([\![a]\!])} \sqsubseteq u_2^{([\![a]\!])}.$$

Thus, inductively, $v_1 < a$ and $v_2^{([\![a]\!])} < b$, so $(v_1, v_2) < c$.

*Case* $D \equiv (\Pi x \in A)B$: In view of Lemma 6.4, it suffices to consider the case

$$\langle w; z \rangle \sqsubseteq h = \bigsqcup_i \langle u_i; v_i \rangle < c.$$

By assumption,

$$(\exists b)(c \to (\lambda x)b \;\&\; (\forall a \in A)(\forall i)(u_i \prec a \text{ implies } v_i^{(\llbracket a \rrbracket)} \prec b(a/x))).$$

To say that $\langle w; z\rangle \sqsubseteq h$ is equivalent to $z \sqsubseteq h(w) = \bigsqcup \{v_i^{(w)} : u_i \sqsubseteq w\}$. To prove $\langle w; z\rangle \prec c$, let $a \in A$ be such that $w \prec a$. For $u_i \sqsubseteq w$ we inductively have $u_i \prec a$ and hence $v_i^{(\llbracket a \rrbracket)} \prec b(a/x)$. Now, $\{v_i^{(\llbracket a \rrbracket)} : u_i \sqsubseteq w\}$ is consistent since $\{u_i : u_i \sqsubseteq w\}$ is consistent and $\bigsqcup_i \langle u_i; v_i\rangle$ exists. Thus it follows from Lemma 6.4 that $\bigsqcup \{v_i^{(\llbracket a \rrbracket)} : u_i \sqsubseteq w\} \prec b(a/x)$. But then we inductively have $z \prec b(a/x)$ and hence $\langle w; z\rangle \prec c$. □

In order to carry out the proof that operational and denotational approximations are identical, we need to extend the operational approximations to judgements with a nontrivial context.

**Definition 6.6.** Suppose $\Gamma \Rightarrow d \in D$, where $\Gamma \equiv x_1 \in A_1, \ldots, x_n \in A_n$. For $\langle u; v\rangle \in \Pi(\llbracket \Gamma \rrbracket, \llbracket D \rrbracket_\Gamma)_c$, where $u = (u_1, \ldots, u_n)$, we say $\langle u; v\rangle$ is an *operational approximation* of $d$, denoted $\langle u; v\rangle \prec d$, if

$$(\forall a_1 \in A_1) \cdots (\forall a_n \in A_n(a_1, \ldots, a_{n-1}/x_1, \ldots, x_{n-1}))[u_1 \prec a_1 \;\&\; u_2^{(\llbracket a_1 \rrbracket)} \prec a_2 \;\&$$
$$\cdots \& \; u_n^{(\llbracket a_1 \rrbracket, \ldots, \llbracket a_{n-1} \rrbracket)} \prec a_n \quad \text{implies} \quad v^{(\llbracket a_1 \rrbracket, \ldots, \llbracket a_n \rrbracket)} \prec d(a_1, \ldots, a_n/x_1, \ldots, x_n)].$$

In addition, if the set $\{\langle u_1; v_1\rangle, \ldots, \langle u_k; v_k\rangle\}$ is consistent in $\Pi(\llbracket \Gamma \rrbracket, \llbracket D \rrbracket_\Gamma)_c$ then $\bigsqcup_i \langle u_i; v_i\rangle \prec d$ if $\langle u_i; v_i\rangle \prec d$ for each $i$.

Recall that the quantifiers in the definition ranges over all $a$ such that $\emptyset \Rightarrow a \in A$. The remaining work lies in the following lemma.

**Lemma 6.7.** *Suppose $\Gamma \Rightarrow d \in D$. If $w \sqsubseteq \llbracket d \rrbracket_\Gamma$ then $w \prec d$.*

In case $\Gamma = \emptyset$ we identify $\langle \bot; v\rangle$ with $v$ and $\llbracket D \rrbracket$ with $\llbracket D \rrbracket_\emptyset(\bot)$. Then we have our main result.

**Theorem 6.8.** *Suppose $\emptyset \Rightarrow d \in D$. Then for $u \in \llbracket D \rrbracket_c$, $u \sqsubseteq \llbracket d \rrbracket$ if and only if $u \prec d$.*

**Proof.** One direction is Lemma 6.2 and the other direction is Lemma 6.7. □

**Corollary 6.9.** *Suppose $\emptyset \Rightarrow d \in D$. Then $d$ terminates if and only if $\llbracket d \rrbracket \neq \bot_{\llbracket D \rrbracket}$.*

**Proof.** By Theorem 6.8 and Lemma 6.3. □

**Proof of Lemma 6.7.** The proof is by induction on $d$. We consider only some of the most interesting cases, leaving the remaining ones to the reader. Throughout we assume $\Gamma \equiv x_1 \in A_1, \ldots, x_n \in A_n$.

*Case* $d \equiv x_i$: Suppose $\langle u; v\rangle \sqsubseteq [\![x_i]\!]_\Gamma$, where $u = (u_1, \ldots, u_n)$. Thus

$$v \sqsubseteq [\![x_i]\!]_\Gamma(u) = u_i \quad \text{in } [\![A_i]\!]_\Gamma(u) = [\![A_i]\!]_{\Gamma_i}(u_1, \ldots, u_{i-1}).$$

Let $a_1 \in A_1, \ldots, a_n \in A_n(a_1, \ldots, a_{n-1}/x_1, \ldots, x_{n-1})$ and suppose

$$u_1 \prec a_1, \ldots, u_i^{([\![a_1]\!], \ldots, [\![a_{i-1}]\!])} \prec a_i, \ldots, u_n^{([\![a_1]\!], \ldots, [\![a_{n-1}]\!])} \prec a_n.$$

We have $v^{([\![a_1]\!], \ldots, [\![a_{i-1}]\!])} = v^{([\![a_1]\!], \ldots, [\![a_n]\!])}$ and, since $v \sqsubseteq u_i$, $v^{([\![a_1]\!], \ldots, [\![a_{i-1}]\!])} \sqsubseteq u_i^{([\![a_1]\!], \ldots, [\![a_{i-1}]\!])}$. By assumption, $u_i^{([\![a_1]\!], \ldots, [\![a_{i-1}]\!])} \prec a_i$, and hence by Lemma 6.5

$$v^{([\![a_1]\!], \ldots, [\![a_n]\!])} = v^{([\![a_1]\!], \ldots, [\![a_{i-1}]\!])} \prec a_i \equiv x_i(a_1, \ldots, a_n/x_1, \ldots, x_n).$$

*Case* $d \equiv (\lambda x)b \in (\Pi x \in A)B$: Suppose $\langle u; v\rangle \sqsubseteq [\![(\lambda x)b]\!]_\Gamma$, that is

$$v \sqsubseteq [\![(\lambda x)b]\!]_\Gamma(u) = \mathrm{curry}([\![b]\!]_{\Gamma, x\in A})(u), \quad \text{where } u = (u_1, \ldots, u_n).$$

Suppose $a_1 \in A_1, \ldots, a_n \in A_n(a_1, \ldots, a_{n-1}/x_1, \ldots, x_{n-1})$ are such that $u_1 \prec a_1, \ldots, u_n^{([\![a_1]\!], \ldots, [\![a_{n-1}]\!])} \prec a_n$. The case when $v = \bot$ is trivial. So suppose $v \neq \bot$. Then

$$v = \langle z; w\rangle \in \Pi([\![A]\!]_\Gamma(y), (y)[\![B]\!]_{\Gamma, x\in A}(u, y))_c.$$

Recall that

$$\langle z; w\rangle^{([\![a_1]\!], \ldots, [\![a_n]\!])} = \langle z'; w'\rangle \quad \text{where}$$
$$z' = z^{([\![a_1]\!], \ldots, [\![a_n]\!])} \quad \text{and} \quad w' = w^{([\![a_1]\!], \ldots, [\![a_n]\!], z')}.$$

Since $\langle z; w\rangle \sqsubseteq [\![(\lambda x)b]\!]_\Gamma(u) = \mathrm{curry}([\![b]\!]_{\Gamma, x\in A})(u)$, we have

$$w \sqsubseteq \mathrm{curry}([\![b]\!]_{\Gamma, x\in A})(u)(z) = [\![b]\!]_{\Gamma, x\in A}(u, z),$$

that is $\langle (u, z); w\rangle \sqsubseteq [\![b]\!]_{\Gamma, x\in A}$.

Inductively, we have $\langle (u, z); w\rangle \prec b$. Let $e \in A(a_1, \ldots, a_n/x_1, \ldots, x_n)$ and suppose $z' = z^{([\![a_1]\!], \ldots, [\![a_n]\!])} \prec e$. Then

$$\begin{aligned} w'' = w^{([\![a_1]\!], \ldots, [\![a_n]\!], [\![e]\!])} &\prec b(a_1, \ldots, a_n, e/x_1, \ldots, x_n, x) \\ &\equiv b(a_1, \ldots, a_n/x_1, \ldots, x_n)(e/x) \end{aligned}$$

Thus $b(a_1, \ldots, a_n, e/x_1, \ldots, x_n, x) \in B(a_1, \ldots, a_n/x_1, \ldots, x_n)(e/x)$, and

$$\begin{aligned} w'' &= w^{([\![a_1]\!], \ldots, [\![a_n]\!], z')([\![a_1]\!], \ldots, [\![a_n]\!], [\![e]\!])} \\ &= w'^{([\![a_1]\!], \ldots, [\![a_n]\!], [\![e]\!])} \\ &= [\![B(a_1, \ldots, a_n/x_1, \ldots, x_n)]\!]^+_{x\in A}[z', [\![e]\!]](w'). \end{aligned}$$

From clause (vi) of Definition 6.1 it follows that

$$\langle z'; w'\rangle \prec (\lambda x)b(a_1, \ldots, a_n/x_1, \ldots, x_n) \equiv ((\lambda x)b)(a_1, \ldots, a_n/x_1, \ldots, x_n)$$

which was to be shown.

*Case* $d \equiv \mathrm{Ap}(c, a) \in B(a/x)$: For simplicity we first consider the case where $\Gamma = \emptyset$. Assume $\mathrm{Ap}(c, a)$ has been obtained from $c \in (\Pi x \in A)B$ and $a \in A$. Let

$\perp \neq w \sqsubseteq [\![\mathrm{Ap}(c, a)]\!]$. Then $[\![c]\!] \neq \perp$ and $[\![\mathrm{Ap}(c, a)]\!] = [\![c]\!]([\![a]\!])$. By the continuity of $[\![c]\!]$ there is $v' \sqsubseteq [\![a]\!]$ such that $w \sqsubseteq ([\![c]\!](v'))^{([\![a]\!])}$, and by the continuity of $[\![B]\!]_{x\in A}$ there is $v'' \sqsubseteq [\![a]\!]$ and $w'' \in [\![B]\!]_{x\in A}(v'')$ such that $w''^{([\![a]\!])} = w$. As usual, we set $\bar{v} = v' \sqcup v''$ and $\bar{w} = w''^{(\bar{v})}$. Then $\bar{v}$ and $\bar{w}$ are compact and $w = \bar{w}^{([\![a]\!])}$. Consider $\langle \bar{v}; \bar{w}\rangle \in \Pi([\![A]\!], [\![B]\!]_{x\in A})_c$. Then we have

$$w \sqsubseteq ([\![c]\!](v'))^{([\![a]\!])} = ([\![c]\!](v'))^{(\bar{v})([\![a]\!])} \sqsubseteq ([\![c]\!](\bar{v}))^{([\![a]\!])}$$

and hence $\bar{w} \sqsubseteq [\![c]\!](\bar{v})$. But then $\langle \bar{v}; \bar{w}\rangle \sqsubseteq [\![c]\!]$, so, inductively, $\langle \bar{v}; \bar{w}\rangle < c$. Furthermore, $\bar{v} \sqsubseteq [\![a]\!]$ and hence, also inductively, $\bar{v} < a$. Note that clause (vi) of Definition 6.1 must hold for $c$ and hence $c \to (\lambda x)b$ and $\bar{w}^{([\![a]\!])} < b(a/x)$. But $c \to (\lambda x)b$ implies $\mathrm{Ap}(c, a) \to b(a/x)$, so $w = \bar{w}^{([\![a]\!])} < \mathrm{Ap}(c, a)$ as was to be shown.

Now consider the case when $\Gamma \neq \emptyset$. Suppose $\langle u; v\rangle \sqsubseteq [\![\mathrm{Ap}(c, a)]\!]_\Gamma$, where $u = (u_1, \ldots, u_n)$, and let $a_1 \in A_1, \ldots, a_n \in A_n(a_1, \ldots, a_{n-1}/x_1, \ldots, x_{n-1})$ be such that $u_1 < a_1, \ldots, u_n^{([\![a_1]\!], \ldots, [\![a_{n-1}]\!])} < a_n$. Assume also that we are in the nontrivial case where $[\![c]\!]_\Gamma(u) \neq \perp$. We need show

$$\begin{aligned}\bar{v} = v^{([\![a_1]\!], \ldots, [\![a_n]\!])} &< \mathrm{Ap}(c, a)(a_1, \ldots, a_n/x_1, \ldots, x_n)\\ &\equiv \mathrm{Ap}(c(a_1, \ldots, a_n/x_1, \ldots, x_n), a(a_1, \ldots, a_n/x_1, \ldots, x_n)).\end{aligned}$$

By assumption, $v \sqsubseteq [\![\mathrm{Ap}(c, a)]\!]_\Gamma(u) = [\![c]\!]_\Gamma(u)([\![a]\!]_\Gamma(u))$. Thus

$$\begin{aligned}\bar{v} &\sqsubseteq [\![\mathrm{Ap}(c, a)]\!]_\Gamma(u)^{([\![a_1]\!], \ldots, [\![a_n]\!])}\\ &\sqsubseteq [\![\mathrm{Ap}(c, a)]\!]_\Gamma([\![a_1]\!], \ldots, [\![a_n]\!])\\ &= [\![\mathrm{Ap}(c, a)(a_1, \ldots, a_n/x_1, \ldots, x_n)]\!]_\emptyset.\end{aligned}$$

By the proof for $\Gamma = \emptyset$, we have

$$\bar{v} < \mathrm{Ap}(c, a)(a_1, \ldots, a_n/x_1, \ldots, x_n)$$

which completes the proof for the general case $\mathrm{Ap}(c, a)$.

*Case* $d \equiv R_\omega(c, (x, y)e) \in C(c/z)$: We consider the case with empty context, leaving the general case to the reader. Suppose $w \sqsubseteq [\![R_\omega(c, (x, y)e)]\!] \in [\![C(c/z)]\!] = [\![C]\!]_{z\in\Omega}([\![c]\!])$. We prove our result distinguishing between the two cases $[\![c]\!] = \omega$ and $[\![c]\!] < \omega$. First consider the case $[\![c]\!] = \omega$. Recall that then

$$[\![R_\omega(c, (x, y)e)]\!] = \bigsqcup_n f_e(n)^{(\omega)}$$

where

$$f_e(0) = \perp_{[\![C]\!]_{z\in\Omega}(0)}, \qquad f_e(n+1) = [\![e]\!]_{x\in\Omega, y\in C(x/z)}(n, f_e(n)).$$

Since $w$ is compact, $w \sqsubseteq f_e(n)^{(\omega)}$ for all sufficiently large $n$. By the continuity of $[\![C]\!]_{z\in\Omega}$ we may choose such an $n$ so that there is $\bar{w} \in [\![C]\!]_{z\in\Omega}(n)_c$ such that $w = \bar{w}^{(\omega)}$. We show how to choose a sequence $v_0, v_1, \ldots, v_n$ such that $v_0 = \perp_{[\![C]\!]_{z\in\Omega}(0)}$, $v_n = \bar{w}$ and for each $i = 0, \ldots, n-1$

$$v_i \sqsubseteq f_e(i) \quad \text{and} \quad v_{i+1} \sqsubseteq [\![e]\!]_{x\in\Omega, y\in C(x/z)}(i, v_i).$$

Note that this means that $\langle (i, v_i); v_{i+1}\rangle \sqsubseteq [\![e]\!]_{x\in\Omega, y\in C(x/z)}$ for $i = 0, \ldots, n-1$. By the choice of $n$, $v_n = \bar{w} \sqsubseteq f_e(n)$. Using the p-continuity of $[\![e]\!] = [\![e]\!]_{x\in\Omega, y\in C(x/z)}$ there is $v_{n-1} \in [\![C]\!]_{z\in\Omega}(n-1)_c$ such that $v_{n-1} \sqsubseteq f_e(n-1)$ and $v_n \sqsubseteq [\![e]\!](n-1, v_{n-1})$. Continuing in this manner we obtain our desired sequence.

By our induction hypothesis we have $i \prec c$ for each $i \in [\![\Omega]\!]_c$. Letting $i > 0$, this means that there is $a_1 \in \Omega$ such that $c \to (a_1)'$ and $i-1 \prec a_1$. It follows that $i \prec a_1$ for each $i \in [\![\Omega]\!]_c$. Continuing in this manner, we obtain a sequence of terms $c \equiv a_0, a_1, \ldots, a_n \in \Omega$ such that $a_i \to (a_{i+1})'$ and $j \prec a_i$ for each $j \in [\![\Omega]\!]_c$. For the sequence $v_0, v_1, \ldots, v_n$ above, we also have by our induction hypothesis that $\langle (i, v_i); v_{i+1}\rangle \prec e$. Now,

$$0 \prec a_n \quad \text{and} \quad v_0{}^{(\omega)} \prec R_\omega(a_n, (x, y)e)$$

by clause (i) of Definition 6.1. It follows that

$$v_1{}^{(\omega)} \prec e(a_n, R_\omega(a_n, (x, y)e)/x, y)$$

since $\langle (0, v_0); v_1\rangle \prec e$. But

$$R_\omega(a_{n-1}, (x, y)e) \to R_\omega((a_n)', (x, y)e) \to e(a_n, R_\omega(a_n, (x, y)e)/x, y)$$

and hence

$$v_1{}^{(\omega)} \prec R_\omega(a_{n-1}, (x, y)e).$$

Continuing in this manner we eventually obtain

$$w = v_n{}^{(\omega)} \prec R_\omega(a_{n-n}, (x, y)e) \equiv R_\omega(c, (x, y)e).$$

This completes the proof when $[\![c]\!] = \omega$.

Now suppose $[\![c]\!] = n \in [\![\Omega]\!]_c$. Then, inductively, $n \prec c$ and hence (in case $n > 0$) there is $a_1 \in \Omega$ such that $c \to (a_1)'$ and $n-1 \prec a_1$. Thus we get a sequence $c \equiv a_0, a_1, \ldots, a_n \in \Omega$ such that $a_i \to (a_{i+1})'$ and $n-i \prec a_i$. By assumption,

$$w \sqsubseteq [\![R_\omega(c, (x, y)e)]\!] = f_e([\![c]\!]) = f_e(n) \in [\![C]\!]_{z\in\Omega}(n).$$

To complete the proof, obtain a sequence $v_0, v_1, \ldots, v_n = w$ as above and prove analogously that

$$v_i \prec R_\omega(a_{n-i}, (x, y)e) \quad \text{for } i = 0, \ldots, n.$$

Then, in particular, $w \prec R_\omega(c, (x, y)e)$. $\square$

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